Available online at www.sciencedirect.com
ScienceDirect
Journal of Functional Analysis 241 (2006) 56–98
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

Linear and bilinear estimates for oscillatory integral operators related to restriction to hypersurfaces ☆

Sanghyuk Lee

Department of Mathematics, University of Wisconsin-Madison, Madison, WI 53706-1388, USA

Received 12 January 2006; accepted 18 May 2006

Available online 18 July 2006

Communicated by Richard B. Melrose

Abstract

We obtain bilinear estimates for oscillatory integral operators which are variable coefficient generalizations of bilinear restriction estimates for hypersurfaces. As applications, we improve the known estimates for oscillatory integrals.

Keywords: Oscillatory integral operators

1. Introduction and the statement of results

In this paper we study the oscillatory integral operator defined by

$$T_\lambda f(z) = \int e^{i\lambda\phi(z,y)} a(z,y) f(y)\, dy, \quad (z,y) \in \mathbb{R}^{n+1} \times \mathbb{R}^n,\ n \geqslant 1,$$

where $a \in C_0^\infty(\mathbb{R}^{2n+1})$ and $\phi \in C^\infty$ on the support of a. The problem we are interested here is to obtain sharp asymptotic decay estimates for $\|T_\lambda\|_{p\to q}$ in terms of λ. The operator T_λ can

☆ This work was supported in part by the Post-doctoral Fellowship Program of Korea Science and Engineering Foundation (KOSEF) and the Research Funds of Seoul National University.
E-mail address: shlee@math.snu.ac.kr.

0022-1236/$ – see front matter
doi:10.1016/j.jfa.2006.05.011

be thought of as a variable coefficient generalization of restriction operator in the sense that the optimal decay estimate

$$\|T_\lambda f\|_q \leqslant C\lambda^{-\frac{n+1}{q}}\|f\|_p \tag{1.1}$$

implies L^p–L^q boundedness of the adjoint of Fourier restriction to the set $\{\nabla_z\phi(z_0, y): |y - y_0| < \epsilon\}$ for a small $\epsilon > 0$ as long as $a(z_0, y_0) \neq 0$ (see [7,12]). Related to Bochner–Riesz conjecture the L^p–L^q boundedness of T_λ was studied with some non-degeneracy conditions on the phase functions [3,7,17]. It is also important to consider homogeneous phases of degree one because these kind of oscillatory integrals naturally appear in the study of wave equations (see [13,16]).

We want to study these oscillatory integral operators by taking so called "bilinear approach," which has already been used for several related problems. Recently, some concrete progress on restriction problems for hypersurfaces has been achieved by making use of this approach [20, 29]. There is a lot of work concerning the restriction to the sphere and other hypersurfaces in $\mathbb{R}^n$. (For the most recent development and related subjects the readers are referred to [2,10,21–25, 27] and further references contained therein.) In comparison with restriction estimates, to obtain (1.1) for general phase one has to handle Kakeya compression phenomenon of tubes defined by curves rather straight lines. Such a curved feature makes the problem more difficult and there are known counterexamples to possible extensions of (1.1) [3,5,26] (also see [11]). The main advantage of bilinear estimates is that one can relax Kakeya compression to obtain the wider range of boundedness than linear estimate by imposing additional separation conditions between two operators. Then, it is possible to improve linear estimate if one can remove the additional conditions.

The aims of this paper are two folds. First, we shall prove variable coefficient generalizations of bilinear restriction estimates to hypersurfaces. Secondly, we apply them to improve the known linear estimates for oscillatory integral operators.

1.1. Bilinear estimates for oscillatory integral operators

For $i = 1, 2$, we define an integral operator by

$$T_i f(z) = \int e^{i\lambda\phi_i(z,\xi)} a_i(z, \xi) f(\xi)\, d\xi, \quad z = (x, t) \in \mathbb{R}^n \times \mathbb{R},$$

where a_i is compactly supported smooth function and ϕ_i is smooth function on the support of a_i. We make several assumptions on the phases ϕ_1, ϕ_2. First,

$$\operatorname{rank} \partial^2_{x\xi}\phi_i = n \tag{1.2}$$

on the support of a_i. So, $\xi \to \partial_x\phi_i(x, t, \xi)$ is diffeomorphism if the support of a_i is sufficiently small. Hence we may assume

$$\partial_t\phi_i(x, t, \xi) = q_i\big(x, t, \partial_x\phi_i(x, t, \xi)\big) \tag{1.3}$$

for some q_i. In fact, $q_i(x, t, \xi) = \partial_t\phi_i(x, t, [\partial_x\phi_i(x, t, \cdot)]^{-1}(\xi))$.

Theorem 1.1. *For $i = 1, 2$, let ϕ_i be smooth function satisfying* (1.2), (1.3). *Suppose that the Hessian matrix $\partial^2_{\xi\xi} q_i$ satisfies $\det \partial^2_{\xi\xi} q_i(z, \partial_x \phi_i(z, \xi_i)) \neq 0$ on the support of a_i and if $(z, \xi_1) \in \operatorname{supp} a_1$ and $(z, \xi_2) \in \operatorname{supp} a_2$,*

$$\left|\left\langle \partial^2_{x\xi} \phi_i(z, \xi_i) \delta(z, \xi_1, \xi_2), \left[\partial^2_{x\xi} \phi_i(z, \xi_i)\right]^{-1} \left[\partial^2_{\xi\xi} q_i(z, u_i)\right]^{-1} \delta(z, \xi_1, \xi_2)\right\rangle\right| \geqslant c > 0 \tag{1.4}$$

for $i = 1, 2$, where $u_i = \partial_x \phi_i(z, \xi_i)$ and $\delta(z, \xi_1, \xi_2) = \partial_\xi q_1(z, u_1) - \partial_\xi q_2(z, u_2)$. Then for any $\epsilon > 0$ there is a constant $C = C(\epsilon)$ such that for $q \geqslant (n+3)/(n+1)$,

$$\|T_1 f T_2 g\|_q \leqslant C \lambda^{-\frac{n+1}{q}+\epsilon} \|f\|_2 \|g\|_2. \tag{1.5}$$

This generalizes bilinear restriction estimates for hypersurfaces with nonvanishing Gaussian curvature [10,20,22,24,25]. The sharp bilinear restrict estimates were first proven by Tao [20] for elliptic surfaces and later these was extended to more general surfaces by the author [10] and Vargas [25]. In case of restriction, we may write $\phi_i(x, t, \xi) = \langle x, \xi\rangle + t\psi_i(\xi)$, $i = 1, 2$. Then the condition (1.4) is equivalent to the one in [10] and (1.5) gives bilinear restriction estimates to surfaces given as graph of ψ_1, ψ_2. If one considers so called elliptic surfaces (e.g., the paraboloid or the sphere [24]), (1.4) is trivially satisfied by simple separation but it becomes different when the surface does not have principal curvatures of the same sign. (See [10,25] for more detail.)

When $n = 1$, Theorem 1.1 is not a new one. It was implicitly used by Hörmander in [7]. Comparing with restriction one can see that Theorem 1.1 is essentially sharp and it seems possible to remove the λ^ϵ-loss for $q > (n+3)/(n+1)$ by adapting the argument in [23] which was used to obtain global estimate from local one. However, we do not intend to do it here.

We also consider oscillatory integral operators defined by homogeneous phases, which are generalizations of the bilinear restriction estimates for the conic surfaces [10,19,29] to variable coefficient versions. These are related to a class of Fourier integral operators studied in [13], which are originated from the study of wave equations.

Theorem 1.2. *Let $n \geqslant 2$ and for $i = 1, 2$, let ϕ_i be a smooth homogeneous function of degree* 1 *in ξ satisfying* (1.2), (1.3), *$\partial_x \phi_i \neq 0$ on the support of a_i. Suppose that the Hessian matrix $\partial^2_{\xi\xi} q_i(z, \partial_x \phi_i(z, \xi_i))$ has maximal rank $n-1$ on $\operatorname{supp} a_i$ and*

$$\left|\left\langle \frac{\partial_x \phi_i(z, \xi_i)}{|\partial_x \phi_i(z, \xi_i)|}, \partial_\xi q_1\bigl(z, \partial_x \phi_1(z, \xi_1)\bigr) - \partial_\xi q_2\bigl(z, \partial_x \phi_2(z, \xi_2)\bigr)\right\rangle\right| \geqslant c > 0 \tag{1.6}$$

for $i = 1, 2$, whenever $(z, \xi_1) \in \operatorname{supp} a_1$ and $(z, \xi_2) \in \operatorname{supp} a_2$. Then for any $\epsilon > 0$, $q \geqslant (n+3)/(n+1)$, (1.5) *holds.*

It is not hard to see that this implies the bilinear restriction estimates for the cone [29] because (1.6) is satisfied as long as the angle between two subsets of the cone is separated by $O(1)$. It also contains the results for the conic surfaces with curvatures of different signs in [10].

1.2. Applications to linear estimates

For restriction estimate, it is relatively easy to obtain linear estimate (1.1) by the rescaling argument in [24] once bilinear one is established. However, for oscillatory integral operators derivation of linear estimate from bilinear estimate becomes more complicated because the phase is no longer linear. The crucial part of rescaling argument is to control uniformly the bounds for the decomposed but the presence of higher-order terms makes it difficult to do so. Even though, Theorems 1.1, 1.2 can be applied to obtain linear estimates if a suitable condition is imposed on the phase.

1.2.1. Hörmander's problem

Related to Bochner–Riesz conjecture, Hörmander [7] considered the problem whether it is possible to obtain the estimate (1.1) for p, q satisfying

$$q > \frac{2n+2}{n}, \qquad \frac{n+2}{q} \leqslant n\left(1 - \frac{1}{p}\right)$$

under the following conditions: on the support of a

$$\operatorname{rank} \partial_{zy}^2 \phi = n, \tag{1.7}$$

and if $\theta \in S^n$ is the unique direction for which $\nabla_y \langle \partial_z \phi, \theta \rangle = 0$, then

$$\det\bigl(\partial_{yy}^2 \langle \partial_z \phi, \theta \rangle\bigr) \neq 0. \tag{1.8}$$

This problem is a natural generalization of restriction to hypersurfaces with nonvanishing Gaussian curvature [7,18]. For $n = 1$, it was proven by Hörmander [7] generalizing the earlier result due to Carleson and Sjölin [6]. In higher dimensions ($n \geqslant 2$), Stein [17] proved it for $q \geqslant (2n+4)/n$. Later, it was shown by Bourgain [3,5] that when $n \geqslant 2$ there are phase functions for which it is impossible to obtain (1.1) for $q < (2n+4)/n$ even though they satisfies (1.7), (1.8). In $\mathbb{R}^3$ he also showed that for a generic phase function (1.1) fails if $q < s$ for some $3 < s < 4$ and a positive result beyond the Stein's result was obtained with a much simpler phase. Some partial improvements were also obtained in [22,24] for special phases. Recently, Wisewell [26] obtained counterexamples which give more concrete range of failure in all dimension bigger than 2 using quadratic phases.

From negative results it is obvious that it is impossible to go beyond the critical $q = (2n+2)/(n-1)$ without an additional assumption. For this we add an elliptic type condition which was already used in [24].

Theorem 1.3. *Suppose that ϕ satisfies (1.7), (1.8) and the Hessian matrix*

$$\partial_{yy}^2 \bigl\langle \partial_z \phi(z_0, y), \theta \bigr\rangle \textit{ has eigenvalues of the same sign} \tag{1.9}$$

on the support of a. Then, for p, q satisfying $q \geqslant 2(n+3)/(n+1)$, $(n+2)/q \leqslant n(1-1/p)$ and $\epsilon > 0$, there is a constant $C = C(\epsilon)$ such that

$$\|T_\lambda f\|_q \leqslant C\lambda^{-\frac{n+1}{q}+\epsilon}\|f\|_p. \tag{1.10}$$

It is well known that $\phi(x,t,y) = (|x-y|^2 + t^2)^{1/2}$ satisfies the conditions (1.7), (1.8) and (1.9) provided a is supported in the set $\{(x,y,t)\colon |x|, |y| \ll 1,\ t \sim 1\}$. Hence, in $\mathbb{R}^{n+1}$ the above theorem gives another proof of the Bochner–Riesz conjecture for $q \geqslant 2(n+3)/(n+1)$ and

$1 \leqslant q \leqslant 2(n+3)/(n+5)$ (see [18]). In [9] it was proven by making use of bilinear restriction estimates for the elliptic surfaces [20]. The counterexamples in [26, Corollary 11] say that in worst case (1.10) is possible only for $q \geqslant 2(n+1)/n + 2/n(2n-1)$ even under the condition (1.9). Especially, when $n = 2$ Theorem 1.3 cannot be extended to any $q < 10/3$. Hence it gives sharp results in $\mathbb{R}^3$ but there is gap in higher dimensions. Theorem 1.3 also implies new estimates for the corresponding curved Kakeya maximal functions (see [3,26]).

1.2.2. Oscillatory integral operators with homogeneous phases

We also try to obtain linear estimate for T_λ defined by homogeneous ϕ with $n \geqslant 2$. In view of the restriction estimates for the conic surfaces which have maximal number of nonvanishing curvatures, it seems natural to ask whether there are estimates (1.1) for

$$q > \frac{2n}{n-1}, \qquad \frac{1}{q} \leqslant \frac{n-1}{n+1}\left(1 - \frac{1}{p}\right)$$

assuming (1.7) and that

$$\operatorname{rank} \partial_{yy}^2 \langle \partial_z \phi, \theta \rangle = n - 1 \tag{1.11}$$

provided $\theta \in S^n$ is the unique direction for which $\nabla_y \langle \partial_z \phi, \theta \rangle = 0$.

This is a natural homogeneous version of the condition in Hörmander's problem, which was used by Mockenhaupt et al. [13] to study the local smoothing properties of a class of Fourier integral operators and they obtained L^2–$L^{(2n+2)/(n-1)}$ estimate for T_λ with optimal decay. Like Hörmander's problem there is a phase function for which (1.1) is no longer valid for $q < 2(n+1)/(n-1)$ when n is odd and $\geqslant 3$. It can be shown by a simple modification of Bourgain's counterexample [3,5]. We briefly explain the case $n = 3$ but without difficulty the argument can be extended to higher dimensions. Let us consider

$$\phi(x, t, y) = xy + 2ty_1y_2/y_3 + t^2y_1^2/y_3$$

and suppose that a is supported in $B(0, \epsilon_0) \times B(e_3, \epsilon_0) \subset \mathbb{R}^4 \times \mathbb{R}^3$, $0 < \epsilon_0 \ll 1$. One can easily see that ϕ satisfies (1.7) and (1.11). Choose $f(y) = e^{i\lambda y_2^2/y_3}$. Then it is not difficult to see that $|T_\lambda f(x,t)| \sim \lambda^{-1/2}$ if $|x_1 - x_2 t| \leqslant c\lambda^{-1}$ and $|x_3 - x_2^2/4| \leqslant c\lambda^{-1}$ for some small $0 < c$. Hence $\|T_\lambda f\|_q \leqslant C\lambda^{-4/q}\|f\|_\infty$ is possible only if $q \geqslant 4$. (See [3,5] for the details.)

This means that we need to impose additional condition on the phase ϕ to extend (1.10) beyond L^2–$L^{(2n+2)/(n-1)}$ estimate. For this purpose we again require that

$$\text{all nonzero eigenvalues of } \partial_{yy}^2 \langle \partial_z \phi, \theta \rangle \text{ have the same sign.} \tag{1.12}$$

Theorem 1.4. *Let $n \geqslant 2$. Suppose that $\phi(z, \cdot)$ is a homogeneous function of degree one and ϕ satisfies* (1.7), (1.11) *and* (1.12) *on the support of a. Then if $n \geqslant 3$, $q \geqslant 2(n+3)/(n+1)$ and $(n+1)/q \leqslant (n-1)(1-1/p)$,* (1.10) *holds. When $n = 2$,* (1.10) *is valid for $q \geqslant 4$, $3/q \leqslant (1 - 1/p)$.*

When $n = 2$, (1.12) is redundant because there is only one nonzero curvature. Hence up to ϵ-loss it gives the optimal result in $\mathbb{R}^3$ which generalizes the restriction estimates for the cone due to Barcelo [1]. In $\mathbb{R}^4$, Theorem 1.4 gives the best possible estimate modulo λ^ϵ-loss which extends Wolff's result [29]. In higher dimensions ($n \geqslant 4$) this gives an improvement over the L^2-result in [13]. When $n \geqslant 4$ it seems possible to show failure of (1.1) for generic phases when $q > s$ with some $s \in (2n/(n-1), 2(n+1)/(n-1))$ adopting arguments in [3,26].

1.2.3. L^p–L^q regularity properties of Fourier integral operators

Theorem 1.2 can be used to obtain L^p–L^q sharp regularity properties of the class Fourier integral operators $I^\mu(Z, Y; \mathcal{C})$ satisfying the curvature condition in [13] and it is equivalent to the cinematic curvature condition defined in [15]. If $\mathcal{F} \in I^{\mu-1/4}(Z, Y; \mathcal{C})$, then it can be written as a finite some of operators defined by

$$\mathcal{F}_\mu f(z) = \int\limits_{\mathbb{R}^n} e^{i\phi(z,\xi)} a(z, \xi) \frac{\widehat{f}(\xi)}{(1+|\xi|^2)^{\mu/2}}\, d\xi, \quad z = (x, t),$$

so that the smooth homogeneous function ϕ satisfies (1.7) and (1.11) and a is a symbol of order zero. The following gives an improvement of the L^2–L^q result in [13].

Corollary 1.5. *Let $\mathcal{F}_\mu$ be given as above. Suppose* $\operatorname{supp} a(\cdot, \xi)$ *is contained in a fixed compact set and suppose that $\phi(z, \cdot)$ is a homogeneous function of degree one and ϕ satisfies* (1.7), (1.11) *and* (1.12). *Then for $2(n^2+2n-1)/(n^2-1) \leqslant q \leqslant \infty$, $(n+1)/q \leqslant (n-1)(1-1/p)$ and $q \geqslant p(n+3)/(n+1)$,*

$$\|\mathcal{F}_\mu f\|_q \leqslant C\|f\|_p \tag{1.13}$$

provided $\mu > 1/p - (n+1)/q + (n-1)/2$.

The estimates for p, q satisfying $(n+1)/q = (n-1)(1-1/p)$ are intermediate ones between Sogge's L^p-local smoothing conjecture [15] and trivial L^1–L^∞ estimate. When $n = 2$, (1.13) was obtained by Schlag and Sogge [14] for the range $1 \leqslant p \leqslant 5/2$ and for $n \geqslant 3$ the optimal L^2–$L^q(2(n+1)/(n-1) \leqslant q \leqslant \infty)$ estimates were shown in [13]. For the solutions of wave equations (i.e. $\phi(x, t, \xi) = \langle x, \xi\rangle + t|\xi|$) it is easy to verify that the condition on μ is sharp (see [23]) and some sharp L^p-local smoothing estimates were obtained by Wolff [28] in $\mathbb{R}^3$, and Łaba and Wolff [8] in higher dimensions. (See [4,23] for earlier partial results.)

Theorems 1.1, 1.2 can serve as bilinear substitutes in variable coefficient setting as the bilinear restriction estimates did in linear problems [23]. Hence these can be applied to variable coefficient generalization of various problems. We hope that it can be done somewhere else.

Throughout this paper C, c stand for constants possibly different at each place. In Section 2, we prove both Theorems 1.1, 1.2 and in Section 3 these are applied to obtained linear estimates (Theorems 1.3, 1.4).

2. Bilinear estimates: proof of Theorems 1.1, 1.2

As in [3,5], we begin with rescaling. For $\lambda \gg 1$, we consider operators

$$\mathcal{L}_i f(x, t) = \int e^{i\phi_i^\lambda(x,t,\xi)} a_i^\lambda(x, t, \xi) f(\xi)\, d\xi, \quad i = 1, 2, \tag{2.1}$$

where

$$\phi_i^\lambda(x,t,\xi)=\lambda\phi_i(x/\lambda,t/\lambda,\xi),\qquad a_i^\lambda(x,t,\xi)=a_i(x/\lambda,t/\lambda,\xi).$$

To prove Theorems 1.1, 1.2, it is needed to show that there is a constant $C=C(\alpha)$ such that

$$\big\|(\mathcal{L}_1 f)(\mathcal{L}_2 g)\big\|_{L^{\frac{n+3}{n+1}}}\leqslant C\lambda^\alpha\|f\|_2\|g\|_2 \tag{2.2}$$

for any $\alpha>0$. Obviously the constant C may also depend on a_1, a_2, ϕ_1, ϕ_2. For this we adapt Wolff's *induction on scales* argument [29] which was used to obtain the optimal bilinear restriction estimate for the cone (also see [10,20,25]). The central part is to establish an iterative estimate which makes it possible to suppress the exponent α as small as possible.

Proposition 2.1. *Suppose* (2.2) *holds for some* $\alpha>0$ *with* $C=\widetilde{C}$, *independent of small smooth perturbation of* ϕ_i, a_i. *Then, for all* $0<\delta,\epsilon\ll 1$, *there is a constant* $C=C(\widetilde{C},\epsilon,\delta,\phi_1,\phi_2,a_1,a_2)$, *independent of* λ, *such that for* $\lambda\gg 1$,

$$\|\mathcal{L}_1 f\mathcal{L}_2 g\|_{L^{\frac{n+3}{n+1}}}\leqslant C\lambda^\epsilon\max\big(\lambda^{\alpha(1-\delta)},\lambda^{c\delta}\big)\|f\|_2\|g\|_2$$

holds for some constant c, *independent of* λ, δ,ϵ, α *and* c, C *are stable under small smooth perturbation of the phase and amplitude functions* ϕ_i, a_i.

There are trivial estimates

$$\big\|(\mathcal{L}_1 f)(\mathcal{L}_2 g)\big\|_{\frac{n+3}{n+1}}\leqslant C\lambda^\alpha\|f\|_2\|g\|_2$$

with large $\alpha=\alpha(n)$. Starting from such a value of α, for any given $\alpha>0$ the bound λ^α can be achieved by iterating Proposition 2.1 finitely many times with suitable choice of δ,ϵ. The constant C in (2.2) may get large by repeated uses of Proposition 2.1 but it is important that c does not. Actually c depends only on n provided ϕ_1, ϕ_2, a_1 and a_2 are uniformly bounded in C^∞ and a_1, a_2 are supported in a fixed compact set.

Theorems 1.1, 1.2 are to be proven by establishing Proposition 2.1 under the different assumptions given in Theorems 1.1, 1.2, respectively. We first give the proof of Theorem 1.1 in Sections 2.1–2.8. One can observe without much difficulty that the same argument works for Theorem 1.2 except for the proof of Lemma 2.7. So, in Section 2.9 we reprove Lemma 2.7 under the different assumptions.

2.1. Decomposition

We decompose the oscillatory integrals into basic functions so that these are essentially supported on a collection of curved tubes but still enjoy certain orthogonality among themselves.

We may assume that

$$\operatorname{supp} a_i\subset X\times T\times \Xi_i\subset\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n,$$

where X, T and Ξ_i are small balls and the assumptions of Theorem 1.1 are satisfied on ϵ_0-neighborhood of $\operatorname{supp} a_i$, $i=1,2$ for some $0<\epsilon_0\ll 1$. By (1.2) it also can be assumed that $\partial_x\phi_i(x,t,\cdot)$, $\partial_\xi\phi_i(\cdot,t,\xi)$ are diffeomorphisms on ϵ_0-neighborhoods of Ξ_i, X, respectively.

For a fixed $\lambda \gg \epsilon_0^{-2}$, we define the spatial grid $\mathcal{X}$ and its dual $\mathcal{X}_i^*$ grid by setting

$$\mathcal{X} = \lambda^{1/2}\mathbb{Z}^n,$$
$$\mathcal{X}_i^* = \big(\Xi_i + O\big(\lambda^{-1/2}\big)\big) \cap \lambda^{-1/2}\mathbb{Z}^n, \quad i = 1, 2,$$

where $\Xi_i + O(\lambda^{-1/2}) = \{\xi\colon \operatorname{dist}(\Xi_i, \xi) \leqslant C\lambda^{-1/2}\}$ and we set

$$W_i = \mathcal{X} \times \mathcal{X}_i^*.$$

Since $x \to \partial_\xi \phi_i(x, t, \xi)$ is a diffeomorphism, for $w = (y, v) \in W_i$ we can define a smooth curve γ_i^w given by

$$\partial_\xi \phi_i\big(\gamma_i^w(t), t, v\big) = -y/\lambda. \tag{2.3}$$

Since $\partial^2_{x\xi}\phi$ is invertible, from direct computation and (1.3) it follows that

$$\frac{d}{dt}\gamma_i^w(t) = -\partial_\xi q_i\big(\gamma_i^w(t), t, \partial_x \phi_i\big(\gamma_i^w(t), t, v\big)\big). \tag{2.4}$$

For $w \in W_i$, we set

$$T_w = \big\{(\lambda x, \lambda t) \in \mathbb{R}^n \times \mathbb{R}\colon |x|, |t| \leqslant C,\ \big|x - \gamma_i^w(t)\big| \leqslant C\lambda^{-1/2}\big\}. \tag{2.5}$$

It is easy to see that for each fixed v, $\{T_{y,v}\}_{y\in\mathcal{X}}$ are essentially disjoint since $\partial_\xi \phi_i(\cdot, t, v)$ is diffeomorphism.

Let η be a smooth function satisfying $\operatorname{supp}\widehat{\eta} \subset B(0, 1) \subset \mathbb{R}^n$ and $\sum_{k\in\mathbb{Z}^n} \eta(\cdot - k) = 1$. Here $B(a, r)$ denotes the open ball centered at a with radius r. Let $\psi \in C_0^\infty(B(0, 1))$ with $\sum_{k\in\mathbb{Z}^n} \psi(\cdot - k) = 1$. For $y \in \mathcal{X}$ and $v \in \mathcal{X}^*$, let us set

$$\eta_y(x) = \eta\big(\lambda^{-1/2}(x - y)\big), \qquad \psi_v(\xi) = \psi\big(\lambda^{1/2}(\xi - v)\big)$$

and for $w = (y, v) \in W_i$,

$$f_w = (\psi_v f) * \mathcal{F}^{-1}(\eta_y),$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform. Then trivially $f = \sum_{w\in W_i} f_w$ if f is supported in $\Xi_i + O(\lambda^{-1/2})$. Hence it follows that

$$\mathcal{L}_i f = \sum_{w\in W_i} \mathcal{L}_i f_w.$$

The following shows that $\mathcal{L}_i f_w$ is essentially supported in the curved tube T_w.

Lemma 2.2. *If $w = (y, v) \in W_i$, then for any N*

$$\big|\mathcal{L}_i f_w(x, t)\big| \leqslant CM\big(\widehat{\psi_v f}(y)\big)\big(1 + \lambda^{1/2}\big|x/\lambda - \gamma_i^{(y,v)}(t/\lambda)\big|\big)^{-N}.$$

Here M is the Hardy–Littlewood maximal function.

Proof. Since $(\psi_v f) * \mathcal{F}^{-1}(\eta_y)$ is supported in $v + O(\lambda^{-1/2})$, we can plug in a harmless smooth function $\widetilde{\psi}_v$ defined by $\widetilde{\psi} \in C_0^\infty(B(0,2))$ in the same way as ψ_v so that $\widetilde{\psi}_v = 1$ on the support of ψ_v. Then, after translation and rescaling we have

$$\mathcal{L}_i f_{(y,v)}(x,t) = \int K_v(x,t,z)\widehat{\psi_v f}(z)\eta\left(\frac{z-y}{\lambda^{1/2}}\right)dz, \tag{2.6}$$

where

$$K_v(x,t,z) = (2\pi)^{-n}\lambda^{-n/2}\int e^{i\Phi(x,t,\xi,z)}a_i^\lambda\big(x,t,\xi/\lambda^{1/2}+v\big)\widetilde{\psi}(\xi)\,d\xi$$

and

$$\Phi(x,t,\xi,z) = \phi_i^\lambda\big(x,t,\xi/\lambda^{1/2}+v\big) + \big(\xi/\lambda^{1/2}+v\big)\cdot z.$$

Here we used the identity $\mathcal{F}^{-1}(\widehat{f_w}) = f_w$.

Note that $|\partial_\xi \Phi(x,t,\xi,z)| \geqslant \lambda^{1/2}|\partial_\xi\phi_i(x/\lambda,t/\lambda,v) + \lambda^{-1}z| - O(1)$ on the support of the integrand. Hence routine integration by parts gives

$$\big|K_v(x,t,z)\big| \leqslant C\lambda^{-n/2}\big(1+\lambda^{1/2}\big|\partial_\xi\phi(x/\lambda,t/\lambda,v)+\lambda^{-1}z\big|\big)^{-N}$$

for any N. Since $x \to \partial_\xi\phi_i(x,t,\xi)$ is invertible, it follows that

$$\big|\partial_\xi\phi_i(x/\lambda,t/\lambda,v) - \partial_\xi\phi_i\big(\gamma_i^{(z,v)}(t/\lambda),t/\lambda,v\big)\big| \sim \big|x/\lambda - \gamma_i^{(z,v)}(t/\lambda)\big|.$$

So, using (2.3) we get

$$\big|K_v(x,t,z)\big| \leqslant C\lambda^{-n/2}\big(1+\lambda^{1/2}\big|x/\lambda-\gamma_i^{(z,v)}(t/\lambda)\big|\big)^{-N}.$$

From (2.3) $|\gamma_i^{(y,v)}(t/\lambda) - \gamma_i^{(y+z,v)}(t/\lambda)| \sim |z|/\lambda$ because $\partial_\xi\phi_i(\cdot,t,\xi)$ is a diffeomorphism. Hence, by translation $z \to y+z$ in (2.6), it is enough to show that for $R, N \gg 1$,

$$\int\left(1+\frac{||a|-|z||}{R}\right)^{-N}R^{-n}\big|\eta(z/R)F(z)\big|\,dz \leqslant C\big(1+|a|/R\big)^{-N}MF(0).$$

(Here $a = x/\lambda - \gamma_i^{(y,v)}(t/\lambda)$.) The above follows from

$$\left(1+\frac{||a|-|z||}{R}\right)^{-N}R^{-n}\big|\eta(z/R)\big| \leqslant C\big(1+|a|/R\big)^{-N}R^{-n}\big(1+|z|/R\big)^{-M}$$

for any M. It is easy to see by considering the cases $|a| \sim |z|$ and $|a| \not\sim |z|$ separately. □

Lemma 2.3. *If* $\operatorname{supp} f \subset \Xi_i$, *then*

$$\lambda^{n/2}\sum_{w=(y,v)\in W_i}\big(M(\widehat{\psi_v f})(y)\big)^2 \leqslant C\int\big|f(y)\big|^2\,dy.$$

Proof. Since $\psi_v f$ is supported in a ball of radius about $\lambda^{-1/2}$, if $|x-x'| \leqslant C\lambda^{1/2}$ $M(\widehat{\psi_v f})(x) \sim M(\widehat{\psi_v f})(x')$. This can be shown using a bump function adapted to the ball where $\psi_v f$ is supported. Since y are separated by $\lambda^{1/2}$,

$$\lambda^{n/2} \sum_{w=(y,v)} \big(M(\widehat{\psi_v f})(y)\big)^2 \leqslant C \sum_v \int \big(M(\widehat{\psi_v f})(x)\big)^2\,dx \leqslant C \sum_v \int |\widehat{\psi_v f}|^2\,dy.$$

The second inequality is from the Hardy–Littlewood maximal theorem. By Plancherel's theorem, the last is bounded by $C\int |f(y)|^2\,dy$. This completes the proof. □

2.2. Reduction

We normalize $\|f\|_2 = \|g\|_2 = 1$ and fix a δ, $0<\delta \ll 1$. We make decomposition

$$\mathcal{L}_1 f = \sum_{w\in W_1} \mathcal{L}_1 f_w, \qquad \mathcal{L}_2 g = \sum_{w\in W_2} \mathcal{L}_2 g_w.$$

Let $Q(0,\lambda) \in \mathbb{R}^{n+1}$ be the cube centered at the origin with side length λ. For Proposition 2.1 it is sufficient to show that (2.2) implies for some $c>0$

$$\Big\| \sum_{w\in W_1} \sum_{w'\in W_2} \mathcal{L}_1 f_w \mathcal{L}_2 g_{w'} \Big\|_{L^{\frac{n+3}{n+1}}(Q(0,\lambda))} \lessapprox \big(\lambda^{\alpha(1-\delta)} + \lambda^{c\delta}\big).$$

Here, $A \lessapprox B$ means there is a constant C_ϵ such that $A \leqslant C_\epsilon \lambda^\epsilon B$ for any $\epsilon>0$, $\lambda \gg 1$. From the proof of Lemma 2.2, discarding harmless $O(\lambda^{-100n})$-terms, we may assume that for a sufficiently small $\tilde\epsilon > 0$

$$W_i = \bigg(\lambda\Big[\bigcup_{(x,t)\in X\times T} \partial_\xi \phi(x,t,\Xi_i) + O(\tilde\epsilon)\Big] \cap \mathcal{X}\bigg) \times \mathcal{X}_i^*.$$

Because $M(\widehat{\psi_v f})(y) \leqslant \lambda^{-n/4}$ from Lemma 2.3 and $\mathcal{L}_i f_w$ is $O(\lambda^{-N})$ for any N if $-y$ is not contained in the set

$$\lambda\Big[\bigcup_{(x,t)\in X\times T} \partial_\xi \phi(x,t,\Xi_i) + O(\tilde\epsilon)\Big], \quad \tilde\epsilon \ll \epsilon_0.$$

Since $\mathcal{L}_1 f_w$, $\mathcal{L}_2 g_{w'}$ are essentially supported on the tubes T_w, $T_{w'}$, respectively, we may assume all these tubes contained in $Q(0, C\lambda)$. Then the number of all relevant w, w' are obviously $O(\lambda^{2n})$.

For each dyadic number h, set

$$W_1(h) = \big\{w=(y,v)\in W_1\colon\ h < M(\widehat{\psi_v f})(y) \leqslant 2h\big\},$$
$$W_2(h) = \big\{w=(y,v)\in W_2\colon\ h < M(\widehat{\psi_v g})(y) \leqslant 2h\big\}.$$

After breaking the sum $\sum_{w\in W_1}\sum_{w'\in W_2}$ into $\sum_{h_1,h_2}\sum_{w\in W_1(h_1)}\sum_{w'\in W_2(h_2)}$, we can discard the terms with $h_1, h_2 \leqslant O(\lambda^{-100n})$ because the contribution from these are negligible. Since we are assuming $\|f\|_2 = \|g\|_2 = 1$, $h_1, h_2 \leqslant O(\lambda^{-n/4})$. Hence, by pigeonholing along dyadic h_1, h_2, it is enough to show that for dyadic h_1 and h_2,

$$\Big\| \sum_{w \in W_1(h_1)} \sum_{w' \in W_2(h_2)} \mathcal{L}_1 f_w \mathcal{L}_2 g_{w'} \Big\|_{L^{\frac{n+3}{n+1}}(Q(0,\lambda))} \lesssim \big(\lambda^{\alpha(1-\delta)} + \lambda^{c\delta}\big).$$

For $w \in W_1(h_1)$ and $w' \in W_2(h_2)$, let us set

$$F_w = \lambda^{-n/4} h_1^{-1} f_w, \qquad G_{w'} = \lambda^{-n/4} h_2^{-1} g_{w'}.$$

From Lemma 2.3, it follows that $\lambda^{n/4} h_1 |W_1(h_1)|^{1/2} \leqslant C$, $\lambda^{n/4} h_2 |W_2(h_2)|^{1/2} \leqslant C$. Hence, it is sufficient to show that for any subset $\mathcal{W}_1 \subset W_1(h_1)$ and $\mathcal{W}_2 \subset W_2(h_2)$,

$$\Big| \sum_{w \in \mathcal{W}_1} \sum_{w' \in \mathcal{W}_2} \mathcal{L}_1(F_w) \mathcal{L}_2(G_{w'}) \Big\|_{L^{\frac{n+3}{n+1}}(Q(0,\lambda))} \lesssim \big(\lambda^{\alpha(1-\delta)} + \lambda^{c\delta}\big) |\mathcal{W}_1|^{1/2} |\mathcal{W}_2|^{1/2}. \tag{2.7}$$

2.3. *Decomposition of $Q(0,\lambda)$ into $\lambda^{1-\delta}$-cubes S*

Now we partition $Q(0,\lambda)$ into essentially disjoint cubes S of side length $\lambda^{1-\delta}$ so that

$$Q(0,\lambda) = \bigcup S$$

and denote by $\mathcal{Q}(\lambda^{1-\delta})$ the collection of these cubes S.

Lemma 2.4. *Suppose* (2.2) *is valid for the phase satisfying conditions in Theorem* 1.1 *with C, independent of λ and stable under small smooth perturbation of ϕ_i, a_i, $i = 1, 2$. Then for $S \in \mathcal{Q}(\lambda^{1-\delta})$,*

$$\big\| \mathcal{L}_1(f) \mathcal{L}_2(g) \big\|_{L^{\frac{n+3}{n+1}}(S)} \leqslant C \lambda^{\alpha(1-\delta)} \|f\|_2 \|g\|_2.$$

Proof. Let (x_0, t_0) be the center of S and set

$$\Phi_i(x,t,\xi) = \lambda^\delta \phi_i\big((x,t)/\lambda^\delta + (x_0,t_0)/\lambda, \xi\big) - \lambda^\delta \phi_i\big((x_0,t_0)/\lambda, \xi\big)$$

and

$$A_i(x,t) = \psi(x,t) a_i\big((x,t)/\lambda^\delta + (x_0,t_0)/\lambda, \xi\big),$$

where ψ is a smooth function supported in $Q(0,2)$ and $\psi = 1$ on $Q(0,1)$. Define

$$\widetilde{\mathcal{L}}_i f = \int e^{i\lambda^{1-\delta} \Phi_i((x,t)/\lambda^{1-\delta}, \xi)} A_i\big((x,t)/\lambda^{1-\delta}, \xi\big) f(\xi)\, d\xi.$$

Since

$$\big\| \mathcal{L}_1(f) \mathcal{L}_2(g) \big\|_{L^{\frac{n+3}{n+1}}(S)} \leqslant \big\| \widetilde{\mathcal{L}}_1(\tilde{f}) \widetilde{\mathcal{L}}_2(\tilde{g}) \big\|_{\frac{n+3}{n+1}}$$

for some $\tilde f, \tilde g$ satisfying $\|f\|_2 = \|\tilde f\|_2$ and $\|g\|_2 = \|\tilde g\|_2$, it is enough to consider $\widetilde{\mathcal{L}}_1 f \widetilde{\mathcal{L}}_2 g$ instead of $\mathcal{L}_1(f)\mathcal{L}_2(g)$. By Taylor's expansion in x, t

$$\Phi_i(x,t) = \big\langle \nabla\phi_i\big((x_0,t_0)/\lambda, \xi\big), (x,t)\big\rangle + \mathcal{E}(x,t,\xi),$$

where $\mathcal{E}$ is a smooth function with $\|\mathcal{E}\|_{C^\infty} = O(\lambda^{-\delta})$. Hence it is not difficult to see that conditions (1.2)–(1.4) are satisfied if λ is sufficiently large. Then by using the assumption (2.2) with λ replaced by $\lambda^{1-\delta}$ we have the required estimate

$$\|\widetilde{\mathcal{L}}_1 f \widetilde{\mathcal{L}}_2 g\|_{\frac{n+3}{n+1}} \leqslant C\lambda^{\alpha(1-\delta)}\|f\|_2\|g\|_2. \qquad \square$$

By localizing to smaller $\lambda^{1-\delta}$-cubes it is possible to obtain a slightly better bound $C\lambda^{\alpha(1-\delta)}$. So, by triangle inequality

$$\Big\|\sum_{w\in\mathcal{W}_1}\sum_{w'\in\mathcal{W}_2}\mathcal{L}_1(F_w)\mathcal{L}_2(G_{w'})\Big\|_{L^{\frac{n+3}{n+1}}(Q(0,\lambda))} \leqslant C\sum_{S\in\mathcal{Q}(\lambda^{1-\delta})}\Big\|\sum_{w\in\mathcal{W}_1}\sum_{w'\in\mathcal{W}_2}\mathcal{L}_1(F_w)\mathcal{L}_2(G_{w'})\Big\|_{L^{\frac{n+3}{n+1}}(S)}.$$

Since $\mathcal{L}_1(F_w)$, $\mathcal{L}_2(G_{w'})$ are essentially supported in tubes T_{w_1}, T_{w_2}, the contribution of $\mathcal{L}_1(F_w)$, $\mathcal{L}_2(G_{w'})$ to the integration on S is negligible if $T_w + O(\lambda^\delta)$ or $T_{w'} + O(\lambda^\delta)$ does not meet S. One might try to apply the induction hypothesis (Lemma 2.4) to each S with this simple observation but it is not enough to sum up the resulting estimates because there are too many w, w' associated to a single S.

To get around it, we use a relation $\approx$ between $w_i \in \mathcal{W}_i$ and $S \in \mathcal{Q}(\lambda^{1-\delta})$ in which overlapping among tubes and cubes is counted in more refined manner. In the next subsection we will define a relation $\approx$ between w_i, $i = 1, 2$ and S satisfying that for all $w_i \in \mathcal{W}_i$

$$\big|\big\{S \in \mathcal{Q}\big(\lambda^{1-\delta}\big):\ w_i \approx S\big\}\big| \lesssim 1. \tag{2.8}$$

Then the right-hand side of the above can be divided into two parts so that

$$\begin{aligned}\Big\|\sum_{w\in\mathcal{W}_1}\sum_{w'\in\mathcal{W}_2}\mathcal{L}_1(F_w)\mathcal{L}_2(G_{w'})\Big\|_{L^{\frac{n+3}{n+1}}(Q(0,\lambda))} &\leqslant \sum_{S\in\mathcal{Q}(\lambda^{1-\delta})}\Big\|\sum_{(w,w')\in\mathcal{W}_1\times\mathcal{W}_2:\ w\approx S,\ w'\approx S}\mathcal{L}_1(F_w)\mathcal{L}_2(G_{w'})\Big\|_{L^{\frac{n+3}{n+1}}(S)} \\ &\quad+ \sum_{S\in\mathcal{Q}(\lambda^{1-\delta})}\Big\|\sum_{(w,w')\in\mathcal{W}_1\times\mathcal{W}_2:\ w\not\approx S \text{ or } w'\not\approx S}\mathcal{L}_1(F_w)\mathcal{L}_2(G_{w'})\Big\|_{L^{\frac{n+3}{n+1}}(S)}.\end{aligned}$$

Roughly, $w_1 \approx S$ means that T_{w_1} is one of highly concentrating tubes on S. As one might expect, the high concentration part is hard to handle directly. However, it is possible to obtain almost optimal estimates for the low concentration part by utilizing orthogonality among wave packets and the geometry of concentrating tubes. Hence, it is enough to get small improvement of bound for the high concentration part. For this, the induction assumption (2.2) is used. Using Lemma 2.4,

$$\sum_{S\in\mathcal{Q}(\lambda^{1-\delta})}\Big\|\sum_{(w,w')\in\mathcal{W}_1\times\mathcal{W}_2:\ w\approx S,\ w'\approx S}\mathcal{L}_1(F_w)\mathcal{L}_2(G_{w'})\Big\|_{L^{\frac{n+3}{n+1}}(S)} \leqslant C\lambda^{\alpha(1-\delta)}\sum_{S\in\mathcal{Q}(\lambda^{1-\delta})}\Big\|\sum_{w\in\mathcal{W}_1(S)}F_w\Big\|_2\Big\|\sum_{w\in\mathcal{W}_2(S)}G_w\Big\|_2, \tag{2.9}$$

where $\mathcal{W}_i(S)=\{w\in\mathcal{W}_i:\ w\approx S\}$, $i=1,2$.

Recalling the definition of F_w and using Plancherel's theorem, we have

$$\Big\|\sum_{w\in\mathcal{W}_1(S)}F_w\Big\|_2^2\leqslant C\lambda^{-n/2}h_1^{-2}\sum_{(y,v)\in\mathcal{W}_1(S)}\big\|(\psi_v f)*\mathcal{F}^{-1}(\eta_y)\big\|_2^2.$$

Using Fourier inversion we observe $\|(\psi_v f)*\mathcal{F}^{-1}(\eta_y)\|_\infty\leqslant C\lambda^{n/2}M(\widehat{\psi_v f})(y)$. Since $(\psi_v f)*\mathcal{F}^{-1}(\eta_y)$ is supported in a ball of radius $\lambda^{-1/2}$ and $M(\widehat{\psi_v f})(y)\sim h_1$ for all $(y,v)\in\mathcal{W}_1$, we see

$$\Big\|\sum_{w\in\mathcal{W}_1(S)}F_w\Big\|_2^2\leqslant C\big|\mathcal{W}_1(S)\big|, \tag{2.10}$$

and similarly

$$\Big\|\sum_{w\in\mathcal{W}_2(S)}G_w\Big\|_2^2\leqslant C\big|\mathcal{W}_2(S)\big|. \tag{2.11}$$

Hence the left-hand side of (2.9) is bounded by

$$C\lambda^{\alpha(1-\delta)}\sum_{S\in\mathcal{Q}(\lambda^{1-\delta})}\big|\{w\in\mathcal{W}_1:\ w\approx S\}\big|^{1/2}\big|\{w\in\mathcal{W}_2:\ w\approx S\}\big|^{1/2}.$$

After applying Schwarz's inequality, changing the order of summation and using (2.8), we see

$$\begin{aligned}\text{the LHS of (2.9)}&\leqslant C\lambda^{\alpha(1-\delta)}\prod_{i=1}^{2}\Big(\sum_{w_i\in\mathcal{W}_i}\big|\big\{S\in\mathcal{Q}\big(\lambda^{1-\delta}\big):\ w_i\approx S\big\}\big|\Big)^{1/2}\\&\lesssim\lambda^{\alpha(1-\delta)}|\mathcal{W}_1|^{1/2}|\mathcal{W}_2|^{1/2}.\end{aligned}$$

Now we are reduced to showing

$$\sum_{S\in\mathcal{Q}(\lambda^{1-\delta})}\Big\|\sum_{(w,w')\in\mathcal{W}_1\times\mathcal{W}_2:\ w\not\approx S\text{ or }w'\not\approx S}\mathcal{L}_1(F_w)\mathcal{L}_2(G_{w'})\Big\|_{L^{\frac{n+3}{n+1}}(S)}\lesssim\lambda^{c\delta}|\mathcal{W}_1|^{1/2}|\mathcal{W}_2|^{1/2}. \tag{2.12}$$

2.4. The relation $\approx$ between $w_i \in \mathcal{W}_i$ and S

The relation $\approx$ here is basically same with the one in [20] except that we are considering curved tube rather than straight one.

Partition $Q(0,\lambda)$ into essentially disjoint cubes q of side length $\lambda^{1/2}$ so that

$$Q(0,\lambda) = \bigcup_{q \in \mathcal{Q}(\lambda^{1/2})} q.$$

We denote by $\mathcal{Q}(\lambda^{1/2})$ the collection of these cubes q. Then we classify $q \in \mathcal{Q}(\lambda^{1/2})$ and $w_i \in \mathcal{W}_i$ according to the degree of overlapping. For $q \in \mathcal{Q}(\lambda^{1/2})$ and $\mathcal{U} \subset \mathcal{W}_i$, set

$$\mathcal{U}(q) = \{ w_i \in \mathcal{U} \colon T_{w_i} \cap \lambda^\delta q \neq \emptyset \}.$$

Here we denote by Cq the cube which has the same center as q and side length C times as long as that of q, that is, $Cq = q + O(\lambda^{1/2})$. For dyadic numbers $1 \leqslant \mu_1, \mu_2 \leqslant \lambda^{100n}$, set

$$\mathcal{Q}(\mu_1,\mu_2) = \{ q \in \mathcal{Q}(\lambda^{1/2}) \colon \mu_1 \leqslant |\mathcal{W}_1(q)| < 2\mu_1,\ \mu_2 \leqslant |\mathcal{W}_2(q)| < 2\mu_2 \}$$

and for $w_i \in \mathcal{W}_i$, let us set

$$\mathcal{Q}(w_i,\mu_1,\mu_2) = \{ q \in \mathcal{Q}(\mu_1,\mu_2) \colon \lambda^\delta q \cap T_{w_i} \neq \emptyset \}$$

and for dyadic numbers $1 \leqslant \nu \leqslant \lambda^{100n}$,

$$\mathcal{W}_i(\nu,\mu_1,\mu_2) = \{ w_i \in \mathcal{W}_i \colon \nu \leqslant |\mathcal{Q}(w_i,\mu_1,\mu_2)| < 2\nu \}.$$

For each dyadic $1 \leqslant \nu,\mu_1,\mu_2 \leqslant \lambda^{100n}$ and $w_i \in \mathcal{W}_i(\nu,\mu_1,\mu_2)$, let $S(w_i,\nu,\mu_1,\mu_2) \in \mathcal{Q}(\lambda^{1-\delta})$ be the cube which maximizes the quantity

$$\left| \{ q \in \mathcal{Q}(\mu_1,\mu_2) \colon q \cap S \neq \emptyset,\ \lambda^\delta q \cap T_{w_i} \neq \emptyset \} \right|.$$

Possibly there may be many candidates for $S(w_i,\nu,\mu_1,\mu_2)$. Then one may simply choose one of them. Since $\#\mathcal{Q}(\lambda^{1-\delta}) \sim \lambda^{(n+1)\delta}$, by averaging over $S \in \mathcal{Q}(\lambda^{1-\delta})$ it follows

$$\left| \{ q \in \mathcal{Q}(\mu_1,\mu_2) \colon q \cap S(w_i,\nu,\mu_1,\mu_2) \neq \emptyset,\ \lambda^\delta q \cap T_{w_i} \neq \emptyset \} \right| \geqslant C\nu\lambda^{-(n+1)\delta} \tag{2.13}$$

because $|\mathcal{Q}(w_i,\mu_1,\mu_2)| \sim \nu$ if $w_i \in \mathcal{W}_i(\nu,\mu_1,\mu_2)$.

We define a relation $\approx$ between w_i and S by saying

$$w_i \approx S \quad \text{if} \quad S \cap 10S(w_i,\nu,\mu_1,\mu_2) \neq \emptyset$$

for any dyadic $1 \leqslant \nu,\mu_1,\mu_2 \leqslant \lambda^{100n}$ and $w_i \in \mathcal{W}_i(\nu,\mu_1,\mu_2)$. Clearly for each w_i, there are $O((\log\lambda)^3)$ cubes S in $\mathcal{Q}(\lambda^{1-\delta})$ for which $w_i \approx S$ since there are $O((\log\lambda)^3)$ dyadic triples (ν,μ_1,μ_2). So (2.8) follows.

2.5. Finer $\lambda^{1/2}$-scale decomposition

Since $|\mathcal{Q}(\lambda^{1-\delta})| \sim \lambda^{(n+1)\delta}$, for (2.12) it suffices to consider a single $S \in \mathcal{Q}(\lambda^{1-\delta})$. That is

$$\Big\| \sum_{w_1 \not\approx S \text{ or } w_2 \not\approx S} \mathcal{L}_1(F_{w_1})\mathcal{L}_2(G_{w_2}) \Big\|_{L^{\frac{n+3}{n+1}}(S)} \lesssim \lambda^{c\delta} |\mathcal{W}_1|^{1/2} |\mathcal{W}_2|^{1/2}.$$

Here, we are assuming that $w_1 \in \mathcal{W}_1$, $w_2 \in \mathcal{W}_2$. Let us recall Hörmander's generalization of Hausdorff–Young's inequality (see [18, p. 377]). Since $\partial^2_{x\xi}\phi_i \neq 0$, by rescaling and integration in t one can easily see

$$\|\mathcal{L}_i f\|_{L^2(S)} \leqslant C\lambda^{(1-\delta)/2} \|f\|_2.$$

By (2.10), (2.11) and Schwarz's inequality,

$$\Big\| \sum_{w_1 \not\approx S \text{ or } w_2 \not\approx S} \mathcal{L}_1(F_{w_1})\mathcal{L}_2(G_{w_2}) \Big\|_{L^1(S)} \lesssim \lambda^{1-\delta} |\mathcal{W}_1|^{1/2} |\mathcal{W}_2|^{1/2}.$$

Hence, in view of interpolation it suffices to show

$$\Big\| \sum_{w_1 \not\approx S \text{ or } w_2 \not\approx S} \mathcal{L}_1(F_{w_1})\mathcal{L}_2(G_{w_2}) \Big\|_{L^2(S)} \lesssim \lambda^{c\delta} \lambda^{-(n-1)/4} |\mathcal{W}_1|^{1/2} |\mathcal{W}_2|^{1/2}.$$

Obviously it follows from

$$\sum_{q \in \mathcal{Q}(\lambda^{1/2}),\ q \subset 2S} \Big\| \sum_{w_1 \not\approx S \text{ or } w_2 \not\approx S} \mathcal{L}_1(F_{w_1})\mathcal{L}_2(G_{w_2}) \Big\|^2_{L^2(q)} \lesssim \lambda^{c\delta} \lambda^{-(n-1)/2} |\mathcal{W}_1| |\mathcal{W}_2|. \tag{2.14}$$

We make several obvious reductions by pigeonholing. From Lemma 2.2 $|\mathcal{L}_1(F_w)|$, $|\mathcal{L}_2(G_w)| \leqslant C\lambda^{-100n}$ on q if $T_{w_i} \cap \lambda^\delta q = \emptyset$. Hence we may replace the inner sum $\sum_{w_1 \not\approx S \text{ or } w_2 \not\approx S}$ by

$$\sum_{(w_1,w_2) \in \mathcal{W}_1(q) \times \mathcal{W}_2(q):\ w_1 \not\approx S \text{ or } w_2 \not\approx S}$$

in the left-hand side of (2.14). Since all the q appearing in (2.14) is contained in $\mathcal{Q}(\mu_1, \mu_2)$ for some dyadic numbers $1 \leqslant \mu_1, \mu_2 \leqslant \lambda^{100n}$, by pigeonholing $\mathcal{Q}(\lambda^{1/2})$ in (2.14) can also be replaced by $\mathcal{Q}(\mu_1, \mu_2)$ for some μ_1, μ_2. We may further replace $\mathcal{W}_1(q)$, $\mathcal{W}_2(q)$ by $\mathcal{W}_1(\nu_1, \mu_1, \mu_2)(q)$, $\mathcal{W}_2(\nu_2, \mu_1, \mu_2)(q)$, respectively, by pigeonholing over dyadic numbers $1 \leqslant \nu_1, \nu_2 \leqslant \lambda^{100n}$. Therefore, the matters are reduced to showing

$$\sum_{q \in \mathcal{Q}(\mu_1,\mu_2),\ q \subset 2S} \Big\| \sum_{(w_1,w_2) \in \widetilde{\mathcal{W}_1}(q) \times \widetilde{\mathcal{W}_2}(q),\ w_1 \not\approx S \text{ or } w_2 \not\approx S} \mathcal{L}_1(F_{w_1})\mathcal{L}_2(G_{w_2}) \Big\|^2_{L^2(q)}$$
$$\lesssim \lambda^{c\delta} \lambda^{-(n-1)/2} |\mathcal{W}_1| |\mathcal{W}_2|,$$

where $\widetilde{\mathcal{W}}_i = \mathcal{W}_i(\nu_i, \mu_1, \mu_2)$, $i = 1, 2$. Note that the condition (1.4) is symmetric. Hence, for (2.14) it is sufficient to show that for any $\mathcal{U}_2 \subset \mathcal{W}_2$ and dyadic numbers $1 \leqslant \nu_1, \mu_1, \mu_2 \leqslant \lambda^{100n}$,

$$\sum_{q \in \mathcal{Q}(\mu_1,\mu_2),\ q \subset 2S} \Bigg\| \sum_{w_1 \in \mathcal{W}_1^{\not\approx S}(\nu_1,\mu_1,\mu_2)(q),\ w_2 \in \mathcal{U}_2(q)} \mathcal{L}_1(F_{w_1}) \mathcal{L}_2(G_{w_2}) \Bigg\|_{L^2(q)}^2$$
$$\lesssim \lambda^{c\delta} \lambda^{-(n-1)/2} |\mathcal{W}_1| |\mathcal{W}_2|, \tag{2.15}$$

where $\mathcal{W}_i^{\not\approx S}(\nu_i, \mu_1, \mu_2)(q) = \{w_i \in \mathcal{W}_i(\nu_i, \mu_1, \mu_2)(q)\colon\ w_i \not\approx S\}$.

2.6. Orthogonality among wave packets

Fix a point $z \in X \times T$ and for $\xi_1 \in \Xi_1$, $\eta_2 \in \Xi_2$, let us define $\Phi^z_{\xi_1,\eta_2} : \Xi_1 \to \mathbb{R}$ by

$$\Phi^z_{\xi_1,\eta_2}(\eta_1) = q_1\big(z, \partial_x \phi_1(z, \xi_1)\big) + q_2\big(z, \big[\partial_x \phi_1(z, \eta_1) + \partial_x \phi_2(z, \eta_2) - \partial_x \phi_1(z, \xi_1)\big]\big)$$
$$- q_1\big(z, \partial_x \phi_1(z, \eta_1)\big) - q_2\big(z, \partial_x \phi_2(z, \eta_2)\big).$$

We also define a set $\Pi^z_{\xi_1,\eta_2}$ by setting

$$\Pi^z_{\xi_1,\eta_2} = \big\{\eta_1 \in \Xi_1\colon\ \Phi^z_{\xi_1,\eta_2}(\eta_1) = 0\big\}.$$

From (1.4), we see

$$\big|\nabla_{\eta_1} \Phi^z_{\xi_1,\eta_2}(\eta_1)\big| = \big|\partial^2_{x\xi} \phi_1(z, \eta_1)\big(\partial_\xi q_2(z, u_2) - \partial_\xi q_1(z, u_1)\big)\big| \geqslant c > 0, \tag{2.16}$$

where $u_1 = \partial_x \phi_1(z, \eta_1)$, $u_2 = \partial_x \phi_1(z, \eta_1) + \partial_x \phi_2(z, \eta_2) - \partial_x \phi_1(z, \xi_1)$. So, dividing the support of a_i into sufficiently small sets, we may assume that the sets $\Pi^z_{\xi_1,\eta_2}$ are smooth hypersurfaces for all $\xi_1 \in \Xi_1$, $\eta_2 \in \Xi_2$ because $\mathcal{L}_1 f \mathcal{L}_2 g$ is written as a finite sum of such operators.

Lemma 2.5. *For $\mathcal{U} \subset \mathcal{W}_i$, define*

$$\mathcal{N}^z(\mathcal{U}) = \sup_{\xi_1,\eta_2} \big|\big\{w = (y, v) \in \mathcal{U}\colon\ v \in \Pi^z_{\xi_1,\eta_2} + O\big(\lambda^{-1/2+\delta}\big)\big\}\big|.$$

(Here the supremum is taken over $\Xi_1 \times \Xi_2$.) Let $q \in \mathcal{Q}(\lambda^{1/2})$ and $c(q)$ be the center of q. If $\mathcal{U}_i \subset \mathcal{W}_i(q)$, $i = 1, 2$, then

$$\Bigg\| \sum_{w_1 \in \mathcal{U}_1} \sum_{w_2 \in \mathcal{U}_2} \mathcal{L}_1(F_{w_1}) \mathcal{L}_2(G_{w_2}) \Bigg\|_{L^2(q)}^2 \lesssim \lambda^{c\delta} \lambda^{-(n-1)/2} \mathcal{N}^{c(q)/\lambda}(\mathcal{U}_1) |\mathcal{U}_1| |\mathcal{U}_2|$$
$$+ C\lambda^{-100n}.$$

Proof. We write

$$\Bigg\| \sum_{w_1 \in \mathcal{U}_1} \sum_{w_2 \in \mathcal{U}_2} \mathcal{L}_1(F_{w_1}) \mathcal{L}_2(G_{w_2}) \Bigg\|_{L^2(q)}^2 = \sum_{w_1 \in \mathcal{U}_1} \sum_{w_2' \in \mathcal{U}_2} \sum_{w_1' \in \mathcal{U}_1} \sum_{w_2 \in \mathcal{U}_2} I_{w_1, w_2', w_1', w_2},$$

where

$$I_{w_1,w_2',w_1',w_2} = \int_q \mathcal{L}_1(F_{w_1})\mathcal{L}_2(G_{w_2})\overline{\mathcal{L}_1(F_{w_1'})\mathcal{L}_2(G_{w_2'})}\,dx\,dt.$$

Let us set $z_0 = c(q)$. We claim that if $w_i = (y_i, v_i)$, $w_i' = (y_i,'' v_i')$, $i = 1, 2$ and

$$\left|\partial_z\phi_1(z_0/\lambda, v_1) + \partial_z\phi_2(z_0/\lambda, v_2) - \partial_z\phi_1(z_0/\lambda, v_1') - \partial_z\phi_2(z_0/\lambda, v_2')\right| \geqslant \lambda^{-\frac{1}{2}+\delta}, \quad (2.17)$$

then $|I_{w_1,w_2',w_1',w_2}| \leqslant C\lambda^{-N}$ for any N. In particular, if $|I_{w_1,w_2',w_1',w_2}| \geqslant C\lambda^{-300n}$, then both of the inequalities

$$|u_1 + u_2 - u_1' - u_2'| \leqslant \lambda^{-\frac{1}{2}+\delta}, \quad (2.18)$$

$$\left|q_1(z_0/\lambda, u_1) + q_2(z_0/\lambda, u_2) - q_1(z_0/\lambda, u_1') - q_2(z_0/\lambda, u_2')\right| \leqslant \lambda^{-\frac{1}{2}+\delta} \quad (2.19)$$

should be satisfied where

$$u_i = \partial_x\phi_i(z_0/\lambda, v_i), \qquad u_i' = \partial_x\phi_i(z_0/\lambda, v_i').$$

Hence if $|\sum_{w_2\in\mathcal{U}_2} I_{w_1,w_2',w_1',w_2}| \geqslant C\lambda^{-200n}$ for some fixed w_1, w_2', w_1', both of (2.18) and (2.19) must be satisfied for some v_2 (equivalently, u_2). Combining these two inequalities we see that if

$$\left|\sum_{w_2\in\mathcal{U}_2} I_{w_1,w_2',w_1',w_2}\right| \geqslant C\lambda^{-200n}$$

for some fixed w_1, w_2', then v_1' satisfies

$$q_1(z_0/\lambda, u_1) + q_2(z_0/\lambda, u_1' + u_2' - u_1) = q_1(z_0/\lambda, u_1') + q_2(z_0/\lambda, u_2') + O\big(\lambda^{-\frac{1}{2}+\delta}\big).$$

From (2.16) it is obvious that v_1' satisfying the above are contained in $\Pi^{z_0/\lambda}_{v_1,v_2'} + O(\lambda^{-1/2+\delta})$. Since $|\mathcal{W}_1|, |\mathcal{W}_2| = O(\lambda^n)$, we can assume that (2.18) and (2.19) are valid for all w_1, w_2, w_1', w_2' by discarding harmless $O(\lambda^{-100n})$-terms. Therefore, it is sufficient to show

$$\sum_{w_1\in\mathcal{U}_1}\sum_{w_2'\in\mathcal{U}_2}\sum_{\{w_1'\in\mathcal{U}_1:\ v_1'\in\Pi^{c(q)/\lambda}_{v_1,v_2'}+O(\lambda^{-1/2+\delta})\}}\Bigg(\sum_{\{w_2\in\mathcal{U}_2:\ u_2=u_1'+u_2'-u_1+O(\lambda^{-1/2+\delta})\}} I_{w_1,w_2',w_1',w_2}\Bigg)$$
$$\lesssim \lambda^{c\delta}\lambda^{-(n-1)/2}\sup_{\xi_1,\eta_2}\left|\left\{w = (y, v)\in\mathcal{U}_1:\ v\in\Pi^z_{\xi_1,\eta_2} + O\big(\lambda^{-1/2+\delta}\big)\right\}\right||\mathcal{U}_1||\mathcal{U}_2|.$$

If w_1, w_2', w_1' are given, then there are at most $O(\lambda^{c\delta})$–v_2 because these are determined by (2.18) and $\partial_x\phi_i(z,\cdot)$ is diffeomorphism. Since all the tubes T_{w_2} meet $\lambda^\delta q$, there are at most $O(\lambda^{c\delta})$–w_2 if v_2 is determined because the tubes T_{y_2,v_2} are essentially disjoint for fixed v_2.

By Lemma 2.2, it is easy to see $|I_{w_1,w_2',w_1',w_2}| \leqslant C\lambda^{-(n-1)/2}$ because the integral is taken over $\lambda^{1/2}$-cube q. Therefore, fixed w_1, w_2', w_1', we get

$$\sum_{\{w_2\in\mathcal{U}_2:\ u_2=u_1'+u_2'-u_1+O(\lambda^{-1/2+\delta})\}} |I_{w_1,w_2',w_1',w_2}| \lesssim \lambda^{c\delta}\lambda^{-(n-1)/2}.$$

This gives the required estimate.

Now it remains to show the claim. Plugging in a harmless smooth function $\eta(\frac{z-z_0}{\lambda^{1/2}})$ adapted to q, we see

$$I_{w_1,w_2',w_1',w_2} = \int K(\xi_1,\xi_2,\eta_1,\eta_2)\big[F_{w_1}(\xi_1)G_{w_2}(\xi_2)F_{w_1'}(\eta_1)G_{w_2'}(\eta_1)\big]\,d\xi_1\,d\xi_2\,d\eta_1\,d\eta_2.$$

Here K is given by

$$K(\xi_1,\xi_2,\eta_1,\eta_2) = \int e^{i\lambda\Psi(z/\lambda,\xi_1,\xi_2,\eta_1,\eta_2)}\eta\left(\frac{z-z_0}{\lambda^{1/2}}\right)dz,$$

where

$$\Psi(z,\xi_1,\xi_2,\eta_1,\eta_2) = \phi_1(z,\xi_1) + \phi_2(z,\xi_2) - \phi_1(z,\eta_1) - \phi_2(z,\eta_2).$$

Since F_{w_1}, $F_{w_1'}$, G_{w_2} and $G_{w_2'}$ are supported in $O(\lambda^{-1/2})$-neighborhood of v_1, v_1', v_2, v_2', respectively, it is sufficient to show that if (2.17) is satisfied, then for any N

$$\big|K(\xi_1,\xi_2,\eta_1,\eta_2)\big| \leqslant C\lambda^{-N}. \tag{2.20}$$

Note that $\xi_1, \xi_2, \eta_1, \eta_2$ are contained in $\lambda^{-1/2}$ neighborhood of v_1, v_1', v_2, v_2', respectively. Hence it is easy to see that if $|z| \leqslant C$

$$\begin{aligned}\lambda\Psi\big(z/\lambda^{1/2} + z_0/\lambda, \xi_1,\xi_2,\eta_1,\eta_2\big) &= \lambda\Psi(z_0/\lambda,\xi_1,\xi_2,\eta_1,\eta_2)\\ &\quad + \lambda^{1/2}\partial_z\Psi(z_0/\lambda, v_1, v_2, v_1', v_2')\cdot z + \mathcal{E}(z,\xi_1,\xi_2,\eta_1,\eta_2),\end{aligned}$$

where $\mathcal{E}(\cdot,\xi_1,\xi_2,\eta_1,\eta_2)$ is in $C^\infty(\{|z|\leqslant C\})$ uniformly. By translation and rescaling

$$K(\xi_1,\xi_2,\eta_1,\eta_2) = \lambda^{\frac{n+1}{2}}\int e^{i\lambda\Psi(z/\lambda^{1/2}+z_0/\lambda,\xi_1,\xi_2,\eta_1,\eta_2)}\eta(z)\,dz.$$

Therefore from (2.17) and routine integration parts, (2.20) follows because

$$\lambda\big|\partial_z\big[\Psi\big(z/\lambda^{1/2}+z_0/\lambda,\xi_1,\xi_2,\eta_1,\eta_2\big)\big]\big| \geqslant C\lambda^{\delta}. \qquad \square$$

2.7. Proof of Theorem 1.1

By showing (2.15), we prove Theorem 1.1. We use the following which will be shown in the next subsection.

Lemma 2.6 *(Combinatorial estimates). For dyadic numbers* $1 \leqslant \mu_1, \mu_2, \nu_1 \leqslant \lambda^{100n}$ *and* $q \in \mathcal{Q}(\mu_1, \mu_2)$, $q \subset 2S$,

$$\mathcal{N}^{c(q)/\lambda}\big(\mathcal{W}_1^{\not\approx S}(\nu_1, \mu_1, \mu_2)(q)\big) \lesssim \lambda^{c\delta} \frac{|\mathcal{W}_2|}{\nu_1 \mu_2},$$

where $c(q)$ *is the center of* q.

From Lemma 2.5, we see

$$\text{LHS of (2.15)} \lesssim \lambda^{c\delta} \lambda^{-\frac{n-1}{2}} \sum_{q \in \mathcal{Q}(\mu_1,\ \mu_2),\ q \subset 2S} \mathcal{N}^{c(q)/\lambda}\big(\mathcal{W}_1^{\not\approx S}(\nu_1, \mu_1, \mu_2)(q)\big) \\ \times \big|\mathcal{W}_1^{\not\approx S}(\nu_1, \mu_1, \mu_2)(q)\big|\big|\mathcal{U}_2(q)\big| + O\big(\lambda^{-50n}\big).$$

Using Lemma 2.6, for (2.15) it suffices to show

$$\frac{|\mathcal{W}_2|}{\nu_1 \mu_2} \sum_{q \in \mathcal{Q}(\mu_1,\ \mu_2),\ q \subset 2S} \big|\mathcal{W}_1^{\not\approx S}(\nu_1, \mu_1, \mu_2)(q)\big|\big|\mathcal{U}_2(q)\big| \lesssim |\mathcal{W}_1||\mathcal{W}_2|.$$

Since $q \in Q(\mu_1, \mu_2)$ and $\mathcal{U}_2 \subset \mathcal{W}_2$, $|\mathcal{U}_2(q)| \leqslant \mu_2$. So we need to show

$$\sum_{q \in \mathcal{Q}(\mu_1,\ \mu_2),\ q \subset 2S} \big|\mathcal{W}_1^{\not\approx S}(\nu_1, \mu_1, \mu_2)(q)\big| \lesssim \nu_1 |\mathcal{W}_1|.$$

Recalling the definition of $\mathcal{W}_1^{\not\approx S}(\nu_1, \mu_1, \mu_2)(q)$ and changing the order of summations, we see that the LHS of the above is bounded by

$$\sum_{w_1 \in \mathcal{W}_1(\nu_1,\ \mu_1,\ \mu_2)} \big|\big\{q \in \mathcal{Q}(\mu_1, \mu_2):\ T_{w_1} \cap \lambda^\delta q \neq \emptyset,\ w_1 \not\approx S\big\}\big|.$$

Since $w_1 \in \mathcal{W}_1(\nu_1, \mu_1, \mu_2)$, $|\{q \in \mathcal{Q}(\mu_1, \mu_2):\ T_{w_1} \cap \lambda^\delta q \neq \emptyset\}| \leqslant \nu_1$. Therefore, we get the required.

2.8. Proof of Lemma 2.6

We need to show that for dyadic numbers $1 \leqslant \mu_1, \mu_2, \nu_1 \leqslant \lambda^{100n}$, $\xi_1 \in \Xi_1$, $\eta_2 \in \Xi_2$, and $q_0 \in \mathcal{Q}(\mu_1, \mu_2)$, $q_0 \subset 2S$,

$$\big|\big\{w = (y, v) \in \mathcal{W}_1^{\not\approx S}(\nu_1, \mu_1, \mu_2)(q_0):\ v \in \Pi_{\xi_1, \eta_2}^{c(q_0)/\lambda} + O\big(\lambda^{-1/2}\big)\big\}\big| \lesssim \lambda^{c\delta} \frac{|\mathcal{W}_2|}{\nu_1 \mu_2}. \quad (2.21)$$

For simplicity we set

$$\mathcal{W}_1^{\not\approx S}\big(\Pi_{\xi_1,\eta_2}^{c(q_0)/\lambda}, q_0\big) = \big\{w = (y, v) \in \mathcal{W}_1^{\not\approx S}(\nu_1, \mu_1, \mu_2)(q_0)\colon\ v \in \Pi_{\xi_1,\eta_2}^{c(q_0)/\lambda} + O\big(\lambda^{-1/2}\big)\big\}.$$

Let $w_1 \in \mathcal{W}_1^{\not\approx S}(\Pi_{\xi_1,\eta_2}^{c(q_0)/\lambda}, q_0)$. Then $T_{w_1} \cap \lambda^\delta q_0 \neq \emptyset$ and $S \cap 10S(w_1, \nu_1, \mu_1, \mu_2) = \emptyset$. Since $q_0 \subset 2S$, $\operatorname{dist}(q_0, 2S(w_1, \nu_1, \mu_1, \mu_2)) \geqslant \lambda^{1-\delta}$. So, by (2.13) we see

$$\big|\big\{q \in \mathcal{Q}(\mu_1, \mu_2)\colon\ \lambda^\delta q \cap T_{w_1} \neq \emptyset, \operatorname{dist}(q_0, q) \geqslant \lambda^{1-\delta}\big\}\big| \gtrsim \nu_1 \lambda^{-(n+1)\delta}.$$

By the definition of $\mathcal{Q}(\mu_1, \mu_2)$, $|\mathcal{W}_2(q)| \sim \mu_2$ for each $q \in \mathcal{Q}(\mu_1, \mu_2)$. By summation in w_1 and w_2, we get

$$\begin{aligned}
&\big|\big\{(q, w_1, w_2) \in \mathcal{Q}(\mu_1, \mu_2) \times \mathcal{W}_1^{\not\approx S}\big(\Pi_{\xi_1,\eta_2}^{c(q_0)/\lambda}, q_0\big) \times \mathcal{W}_2\colon\\
&\qquad \lambda^\delta q \cap T_{w_1} \neq \emptyset, \operatorname{dist}(q_0, q) \geqslant \lambda^{1-\delta}, \lambda^\delta q \cap T_{w_2} \neq \emptyset\big\}\big|\\
&\quad \gtrsim \lambda^{-c\delta} \nu_1 \mu_2 \big|\mathcal{W}_1^{\not\approx S}\big(\Pi_{\xi_1,\eta_2}^{c(q_0)/\lambda}, q_0\big)\big|.
\end{aligned} \tag{2.22}$$

Lemma 2.7. *For each $w_2 \in \mathcal{W}_2$, set*

$$\begin{aligned}
\Delta = \big\{&(q, w_1) \in \mathcal{Q}(\mu_1, \mu_2) \times \mathcal{W}_1^{\not\approx S}\big(\Pi_{\xi_1,\eta_2}^{c(q_0)/\lambda}, q_0\big)\colon\\
&\lambda^\delta q \cap T_{w_1} \neq \emptyset, \operatorname{dist}(q_0, q) \geqslant \lambda^{1-\delta}, \lambda^\delta q \cap T_{w_2} \neq \emptyset\big\}.
\end{aligned}$$

Then, $|\Delta| \leqslant C\lambda^{c\delta}$ for some $c > 0$, independent of λ and w_2.

Using Lemma 2.7, the LHS of (2.22) is bounded by $\lambda^{c\delta}|\mathcal{W}_2|$. Therefore,

$$\nu_1 \mu_2 \big|\mathcal{W}_1^{\not\approx S}\big(\Pi_{\xi_1,\eta_2}^{c(q_0)/\lambda}, q_0\big)\big| \lesssim \lambda^{c\delta} |\mathcal{W}_2|.$$

From this (2.21) follows. It remains to show Lemma 2.7.

Proof of Lemma 2.7. We may assume that the sets Ξ_1, Ξ_2 are small enough so that $\partial_\xi q_1(x, t, \cdot)$, $\partial_\xi q_2(x, t, \cdot)$ are diffeomorphisms on Ξ_1, Ξ_2, respectively, since $\det(\partial_{\xi\xi}^2 q_1)$, $\det(\partial_{\xi\xi}^2 q_2) \neq 0$.

Let us set $\mathcal{Q}(\Delta) = \{q\colon\ (q, w_1) \in \Delta \text{ for some } w_1\}$. First, we claim that

$$\big|\mathcal{Q}(\Delta)\big| \leqslant C\lambda^{c\delta} \tag{2.23}$$

for some $c > 0$. Assuming this for a moment we prove Lemma 2.7. By (2.23) it is enough to show that for each $q \in \mathcal{Q}(\Delta)$,

$$\big|\big\{w_1\colon\ (q, w_1) \in \Delta\big\}\big| \leqslant C\lambda^{c\delta}. \tag{2.24}$$

Recalling that $c(q_0)$ is the center of q_0, we set $(x_0, t_0) = c(q_0)/\lambda$. For any set A and $\nu > 0$, let us set

$$D_\nu A = \{\nu a\colon\ a \in A\}.$$

From the definition of T_{w_1} any tube $D_{\lambda^{-1}} T_{w_1}$ is contained in a $O(\lambda^{-1/2+\delta})$-neighborhood of a curve $(\gamma(v,t),t)$ for some $v \in \mathcal{X}_1^*$ which is given by

$$\frac{d}{dt}\gamma(v,t) = -\partial_\xi q_1\big(\gamma(v,t),t,\partial_x\phi_1\big(\gamma(v,t),t,v\big)\big), \qquad \gamma(v,t_0)=x_0, \tag{2.25}$$

because T_{w_1} meets $\lambda^\delta q_0$.

Fix a $q \in \mathcal{Q}(\Delta)$ and let (x_q,t_q) be the center of the cube q/λ. Since T_{w_1} intersects $\lambda^{c\delta} q_0$ if $w_1=(y_1,v_1) \in \mathcal{W}_1^{\not\approx S}(\Pi_{\xi_1,\eta_2}^{(x_0,t_0)}, q_0)$, for each v_1 there are at most $O(\lambda^{c\delta})$ of y_1 such that $(q,w_1) \in \Delta$. Hence, for (2.24) we need to show that there are at most $O(\lambda^{c\delta})$ of v in $\mathcal{X}_1^*$ satisfying

$$\big|\gamma(v,t_q) - x_q\big| \leqslant C\lambda^{\delta-1/2}. \tag{2.26}$$

Indeed, by Taylor's expansion

$$\gamma(v,t) = x_0 + (t-t_0)\partial_\xi q_1\big(x_0,t_0,\partial_x\phi_1(x_0,t_0,v)\big) + (t-t_0)^2\eta(t,v), \tag{2.27}$$

where η is a smooth function. Since $\partial_\xi q_1(x_0,t_0,\cdot)$ and $\partial_x\phi_1(x_0,t_0,\cdot)$ are diffeomorphisms, it follows that

$$\big|\gamma(v,t_q) - \gamma(u,t_q)\big| \geqslant C|t_q - t_0||u-v|.$$

Since the distance between q and q_0 is $\geqslant C\lambda^{1-\delta}$, $|(x_0,t_0)-(x_q,t_q)| \geqslant C\lambda^{-\delta}$. It is easy to see that $|t_q - t_0| \geqslant C\lambda^{-\delta}$ because $\lambda^\delta q$ cannot meet any T_{w_1} if $|x_0 - x_q| \geqslant C|t_0 - t_q|$ for some large C. Hence v satisfying (2.26) should be contained in a ball of radius $C\lambda^{2\delta-1/2}$. Therefore we get (2.24).

Now we prove (2.23). Since any q in Δ must intersect $T_{w_1} + O(\lambda^{1/2+\delta})$ for some w_1, it is obvious that if $(q,w_1) \in \Delta$ for some w_1,

$$q \subset \bigcup_{\{w_1=(y,v)\in\mathcal{W}_1(q_0):\ v\in\Pi_{\xi_1,\eta_2}^{(x_0,t_0)}+O(\lambda^{-1/2+\delta})\}} \big(T_{w_1} + O\big(\lambda^{1/2+\delta}\big)\big). \tag{2.28}$$

The set on the RHS of the above is essentially $\lambda^{1/2+\delta}$-neighborhood of a cone-like set having $\lambda(x_0,t_0)$ as its focusing point. Indeed, for small $\epsilon > 0$, let us set

$$\Gamma^{(x_0,t_0)} = \bigcup_{v\in\Pi_{\xi_1,\eta_2}^{(x_0,t_0)}} \big\{\big(\gamma(v,t),t\big):\ \lambda^{-\delta}/2 \leqslant |t-t_0| \leqslant \epsilon\big\}.$$

From (2.28) we see if $q \in \mathcal{Q}(\Delta)$, then $q \subset D_\lambda\Gamma^{(x_0,t_0)} + O(\lambda^{1/2+\delta})$ because $\mathrm{dist}(q_0,q) \geqslant \lambda^{1-\delta}$. Furthermore, if $(q,w_1) \in \Delta$ for some w_1, then $q \subset (T_{w_2} + O(\lambda^{1/2+\delta})) \cap (D_\lambda\Gamma^{(x_0,t_0)} + O(\lambda^{1/2+\delta}))$. Hence, we see

$$\bigcup_{q\in\mathcal{Q}(\Delta)} q \subset \big(T_{w_2} + O\big(\lambda^{1/2+\delta}\big)\big) \cap \big(D_\lambda\Gamma^{(x_0,t_0)} + O\big(\lambda^{1/2+\delta}\big)\big).$$

Since q are disjoint $\lambda^{1/2}$-cubes, for (2.23) it is enough to show that

$$\big(T_{w_2}+O\big(\lambda^{1/2+\delta}\big)\big)\cap\big(D_\lambda\Gamma^{(x_0,t_0)}+O\big(\lambda^{1/2+\delta}\big)\big)\subset B\big(p,C\lambda^{1/2+\delta}\big) \tag{2.29}$$

for some $p\in\mathbb{R}^{n+1}$. For this it is sufficient to show that T_{w_2} intersects $D_\lambda\Gamma^{(x_0,t_0)}$ transversally whenever T_{w_2} intersects $D_\lambda\Gamma^{(x_0,t_0)}$. Let us consider the map defined by

$$\Pi^{(x_0,t_0)}_{\xi_1,\eta_2}\times(-\epsilon,\epsilon):(v,t)\to\big(\gamma(v,t),t\big),$$

where $\gamma(v,\cdot)$ is given by (2.25).

Let $z_1=(\gamma(v_1,t_1),t_1)\in\Gamma^{(x_0,t_0)}$ for some t_1 and $v_1\in\Pi^{(x_0,t_0)}_{\xi_1,\eta_2}$. Let $V_1,V_2,\dots,V_{n-1}$ be an orthogonal normal basis for the tangent space $T_{v_1}(\Pi^{(x_0,t_0)}_{\xi_1,\eta_2})$ of $\Pi^{(x_0,t_0)}_{\xi_1,\eta_2}$ at v_1. Since $\Gamma^{(x_0,t_0)}$ is contained the image of the map in the above, the tangent space of $\Gamma^{(x_0,t_0)}$ at z_1 is spanned by the vectors

$$\big(\partial_v\gamma(v_1,t_1)V_1,0\big),\dots,\big(\partial_v\gamma(v_1,t_1)V_{n-1},0\big),\big(\partial_\xi q_1\big(z_1,\partial_x\phi_1(z_1,v_1)\big),1\big).$$

From (2.27) the vectors $\partial_v\gamma(v_1,t_1)V_i$, $i=1,\dots,n-1$ are parallel to

$$\partial^2_{\xi\xi}q_1(U_0)\partial^2_{x\xi}\phi_1(x_0,t_0,v_1)V_i+O\big(|t_1-t_0|\big),\quad i=1,\dots,n-1,$$

respectively, where $U_0=(x_0,t_0,\partial_x\phi_1(x_0,t_0,v_1))$. Note the lengths of all the vectors are ~ 1 because both $\partial^2_{\xi\xi}q_1$, $\partial^2_{x\xi}\phi_1$ are invertible and $|t_1-t_0|$ is small. On the other hand, all the tangent vectors to T_{w_2} are parallel to $(\partial_\xi q_2(z_2,\partial_x\phi_2(z_2,v_2)),1)$ for some z_2 and $v_2\in\mathcal{X}^*_2$. Let $M=\partial^2_{\xi\xi}q_1(U_0)\partial^2_{x\xi}\phi_1(x_0,t_0,v_1)$. Then, it is obvious that whenever T_{w_2} intersects $D_\lambda\Gamma^{(x_0,t_0)}$ T_{w_2} meets $D_\lambda\Gamma^{(x_0,t_0)}$ transversally if

$$\mathrm{dist}\big(\big(\partial_\xi q_1(U_1)-\partial_\xi q_2(U_2)\big),\mathrm{span}\{MV_1,\dots,MV_{n-1}\}\big)\neq 0, \tag{2.30}$$

where $U_i=(z_i,(\partial_x\phi_i(z_i,v_i)))$, $i=1,2$. Since M is invertible, (2.30) is equivalent to $\mathrm{dist}(M^{-1}(\partial_\xi q_1(U_1)-\partial_\xi q_2(U_2)),T_{v_1}(\Pi^{(x_0,t_0)}_{\xi_1,\eta_2}))\neq 0$. By (2.16) the normal vector of $T_{v_1}(\Pi^{(x_0,t_0)}_{\xi_1,\eta_2})$ is parallel to $\pm\partial^2_{x\xi}\phi_1(x_0,t_0,v_1)(\partial_\xi q_1(U_1')-\partial_\xi q_2(U_2'))$ for some U_1' and U_2'. Hence (2.30) follows if we can show

$$\big|\big\langle\partial^2_{x\xi}\phi_1(x_0,t_0,v_1)\big(\partial_\xi q_1(U_1')-\partial_\xi q_2(U_2')\big),M^{-1}\big(\partial_\xi q_1(U_1)-\partial_\xi q_2(U_2)\big)\big\rangle\big|\geqslant c>0.$$

This is easy to see by (1.4) and continuity because we may assume that the support of a_1 and a_2 are sufficiently small. □

2.9. Proof of Theorem 1.2

Without difficulty one can check that the decomposition in Section 2.1 does not depend on particular curvature conditions for the phases as long as (1.2) is satisfied. Especially, Lemma 2.2 remain valid. Using the same decomposition, it is not hard to see that the remaining arguments for the proof of Theorem 1.1 (Sections 2.2–2.8) work under the assumption of Theorem 1.2 except for the proof of Lemma 2.7. Since $\partial_\xi q_1$, $\partial_\xi q_2$ are no longer diffeomorphisms, the previous proof does not work any more. But it can be salvaged by making use of homogeneity of the phases ϕ_1 and ϕ_2. Here we reprove Lemma 2.7 under the assumptions of Theorem 1.2.

Transversality between T_{w_2} and collection of $\{T_{w_1}\}$

The procedure of the proof is similar to that of Lemma 2.7 and we keep the same notations as in Section 2.8. Define for small $\epsilon > 0$

$$\Gamma^{(x_0,t_0)} = \bigcup_{v\in\Xi_1} \big\{\big(\gamma(v,t_q), t+t_0\big)\colon \lambda^{-\delta}/2 \leqslant |t| \leqslant \epsilon\big\},$$

where $\lambda(x_0,t_0)$ is the center of q_0 and $\gamma(v,\cdot)$ is given by (2.25). Here we use Ξ_1 instead of $\Pi^{c(q_0)/\lambda}_{\xi_1,\eta_2}$. Because of homogeneity of q_i, $\Gamma^{(x_0,t_0)}$ is already a conical set without any restrictions on v.

Let $\mathcal{Q}(\Delta)$ be the same one as in the proof of Lemma 2.7. Then, any $q \in Q(\Delta)$ is contained in $D_\lambda\Gamma^{(x_0,t_0)} + O(\lambda^{1/2+\delta})$. Hence

$$\mathcal{Q}(\Delta) \subset \big(T_{w_2} + O\big(\lambda^{1/2+\delta}\big)\big) \cap \big(D_\lambda\Gamma^{(x_0,t_0)} + O\big(\lambda^{1/2+\delta}\big)\big)$$

because $\operatorname{dist}(q_0,q) \geqslant \lambda^{1-\delta}$ if $q \in \mathcal{Q}(\Delta)$. We claim (2.29) again. Then, from the above we get

$$\big|\mathcal{Q}(\Delta)\big| \leqslant C\lambda^{c\delta}.$$

To show (2.29), as before it is sufficient to show that T_{w_2} intersects $D_\lambda\Gamma^{(x_0,t_0)}$ transversally whenever it intersects $D_\lambda\Gamma^{(x_0,t_0)}$. Consider

$$\Xi_1 \times (-\epsilon,\epsilon)\colon (v,t) \to \big(\gamma(v,t),t\big),$$

where $\gamma(v,\cdot)$ is given by (2.25).

Let $z_1 = (\gamma(v_1,t_1),t_1) \in \Gamma^{(x_0,t_0)}$ for some v_1,t_1. Then the tangent space of $\Gamma^{(x_0,t_0)}$ at z_1 is spanned by the vectors

$$\begin{gathered}\big(\partial_v\gamma(v_1,t_1)e_1,0\big),\quad \big(\partial_v\gamma(v_1,t_1)e_2,0\big),\quad \ldots,\quad \big(\partial_v\gamma(v_1,t_1)e_n,0\big),\\ \big(\partial_\xi q_1\big(z_1,\partial_x\phi_1(z_1,v_1)\big),1\big),\end{gathered}$$

where $e_1,\ldots,e_n$ are the standard basis for $\mathbb{R}^n$. On the other hand, the tangent vector to T_{w_2} is parallel to $(\partial_\xi q_2(z_2,\partial_x\phi_2(z_2,v_2)),1)$ for some z_2,v_2. Note that $|\partial_\xi q_1 - \partial_\xi q_2| \sim 1$. Hence $D_\lambda\Gamma^{(x_0,t_0)}$ meets T_{w_2} transversally whenever it meets T_{w_2} if

$$\operatorname{dist}\big(\partial_v\gamma(v_1,t_1)\big[\mathbb{R}^n\big], (\partial_\xi q_1 - \partial_\xi q_2)\big) \neq 0.$$

Here $\partial_v\gamma(v_1,t_1)[\mathbb{R}^n]$ is the image of $\mathbb{R}^n$ under $\partial_v\gamma(v_1,t_1)$. By (2.27), it is easy to see that the basis for $\partial_v\gamma(v_1,t_1)[\mathbb{R}^n]$ are contained in

$$\partial^2_{\xi\xi}q_1\big(x_0,t_0,\partial_x\phi_1(x_0,t_0,v_1)\big)\partial^2_{x\xi}\phi_1(x_0,t_0,v_1)\big[\mathbb{R}^n\big] + O\big(|t_0-t_1|\big).$$

Since $\partial_\xi q_1 - \partial_\xi q_2 \neq 0$ from (1.6), $|t_0 - t_1| < \epsilon$ and $\partial^2_{x\xi}\phi_1$ is invertible, transversality follows from

$$\mathrm{dist}\big((\partial_\xi q_1 - \partial_\xi q_2), \partial^2_{\xi\xi} q_1\big(x_0, t_0, \partial_x \phi_1(x_0, t_0, v_1)\big)\big[\mathbb{R}^n\big]\big) \geqslant c > 0.$$

Since $\partial^2_{\xi\xi} q_1$ has rank $n-1$ and $\partial_\xi q_1(x_0, t_0, \cdot)$ is homogeneous, the null space of the matrix $\partial^2_{\xi\xi} q_1(x_0, t_0, \partial_x\phi_1(x_0, t_0, v_1))$ is spanned by $\partial_x\phi_1(x_0, t_0, v_1)$ itself. Furthermore, $\partial_x\phi_1(x_0, t_0, v_1)$ is perpendicular to the range of matrix $\partial^2_{\xi\xi} q_1$ at $(x_0, t_0, \partial_x\phi_1(x_0, t_0, v_1))$. Hence the above condition is equivalent to

$$\left\langle \frac{\partial_x\phi_1(x_0, t_0, v_1)}{|\partial_x\phi_1(x_0, t_0, v_1)|}, \partial_\xi q_1 - \partial_\xi q_2 \right\rangle \neq 0$$

which is a consequence of (1.6) and continuity because we can assume that the support a_i are sufficiently small. So, T_{w_2} meets $D_\lambda \Gamma^{(x_0,t_0)}$ transversally if it does. Therefore (2.29) follows.

Transversality between null direction and $\Pi^{(x_0,t_0)}_{\xi_1,\eta_2}$

Since $|\mathcal{Q}(\Delta)| \leqslant C\lambda^{c\delta}$, it is enough to show that for each $q \in \mathcal{Q}(\Delta)$ there are at most $O(\lambda^{c\delta})$ of w_1 such that $(w_1, q) \in \Delta$.

Let $q \in \mathcal{Q}(\Delta)$ and let $\lambda(x_q, t_q)$ be the center of q. Since T_{w_1} intersects $\lambda^{c\delta} q_0$ if $(w_1, q) \in \Delta$, it is enough to show that there are at most $O(\lambda^\delta)$ of $v_1 \in \mathcal{X}^*$ satisfying

$$\gamma(v, t_q) = x_q + O\big(\lambda^{-1/2+\delta}\big), \qquad v_1 \in \Pi^{(x_0,t_0)}_{\xi_1,\eta_2} + O\big(\lambda^{-1/2+\delta}\big).$$

Since $\xi \to \partial_x\phi_1(x, t, \xi)$ is invertible, it is sufficient to show that there are at most $O(\lambda^{c\delta})$ of $u_1 = \partial_x\phi_1(x_0, t_0, v_1)$ satisfying

$$\begin{aligned} &u_1 \in \widetilde{\Pi}^{(x_0,t_0)}_{\xi_1,\eta_2} + O\big(\lambda^{-1/2+\delta}\big), \\ &\tilde{\gamma}(u_1, t_q) = \gamma\big(\big[\partial_x\phi(x_0, t_0, \cdot)\big]^{-1}(u_1), t_q\big) = x_q + O\big(\lambda^{-1/2+\delta}\big), \end{aligned} \tag{2.31}$$

where $\widetilde{\Pi}^{(x_0,t_0)}_{\xi_1,\eta_2}$ is the set

$$\big\{u_1 \in \partial_x\phi_1(z_0, \Xi_1)\colon\ q_1(z_0, u_1') + q_2(z_0, u_1 + u_2 - u_1') = q_1(z_0, u_1) + q_2(z_0, u_2)\big\}$$

with $z_0 = (x_0, t_0)$, $u_1' = \partial_x\phi_1(x_0, t_0, \xi_1)$ and $u_2 = \partial_x\phi_2(x_0, t_0, \eta_2)$.

Since $\partial_\xi q_1(x_0, t_0, \cdot)$ is a homogeneous function of degree zero, obviously its null direction at u_1 is u_1. This means that it is the only direction along which $\partial_\xi q_1(x_0, t_0, \cdot)$ fails to be one-to-one because $\partial^2_{\xi\xi} q_1$ has rank $n-1$. Note that the normal vector of $\widetilde{\Pi}^{(x_0,t_0)}_{\xi_1,\eta_2}$ is parallel to $\partial_\xi q_1 - \partial_\xi q_2$. Hence, from (1.6) and continuity it follows that

$$\big|\big\langle u_1/|u_1|, \partial_\xi q_1 - \partial_\xi q_2\big\rangle\big| \geqslant c > 0$$

because we may assume that the supports of a_1, a_2 are sufficiently small. This implies that the null direction $u_1 = \partial_x \phi_1(x_0, t_0, v_1)$ is transversal to $\widetilde{\Pi}^{(x_0,t_0)}_{\xi_1,\eta_2}$. Therefore we can see that the function

$$\partial_\xi q_1(x_0, t_0, \cdot)\big|_{\widetilde{\Pi}^{(x_0,t_0)}_{\xi_1,\eta_2}}$$

is one-to-one because $\partial^2_{\xi\xi} q_1$ has rank $n-1$. From (2.27) we have $\tilde{\gamma}(u, t_q) = x_0 + (t_q - t_0) \times \partial_\xi q_1(x_0, t_0, u) + \frac{1}{2}(t_q - t_0)^2 \tilde{\eta}(t, u)$. Hence,

$$\big|\tilde{\gamma}(u, t_q) - \tilde{\gamma}(u', t_q)\big| \geqslant C|t_q - t_0||u - u'|, \quad \text{if } u, u' \in \widetilde{\Pi}^{(x_0,t_0)}_{\xi_1,\eta_2}.$$

Since $|t_q - t_0| \geqslant \lambda^{-\delta}$, u_1 satisfying (2.31) must be contained in a ball of radius $C\lambda^{-1/2+2\delta}$. There are at most $O(\lambda^{c\delta})$ u_1 satisfying (2.31) because $\{u_1 = \partial_x \phi_1(x_0, t_0, v_1)\colon\ v_1 \in \mathcal{X}^*\}$ is a $C\lambda^{-1/2}$-separated set. This completes the proof.

3. Application to linear estimates: proof of Theorems 1.3 and 1.4

In this section we prove linear estimates (Theorems 1.3, 1.4). We adapt the rescaling argument in [24] (also see [10,22,23,29]). Unlike the restriction case, simple rescaling does not work in our case because the phase function may contain higher order nonlinear terms. Major problem is to obtain a uniform bound for operators resulted from decomposition. To achieve this, we make further decomposition in spatial variable z to utilize a localization property of the oscillatory integral.

3.1. *Proof of Theorem 1.3*

We begin with finding a normalization of the phase function by smooth changes of variables. We write $z = (x, t) \in \mathbb{R}^n \times \mathbb{R}$. By translation we may assume that a is supported in a small neighborhood of the origin and θ in (1.8) is parallel to t-axis. To say, $\partial_t(\partial_y \phi)(0) = 0$ and $\partial^2_{xy}\phi(0)$ is an invertible matrix.

Lemma 3.1. *If ϕ satisfies the conditions* (1.7) *and* (1.8) *with θ parallel to t-axis on a small neighborhood of the origin, then by a smooth change of variables we can assume*

$$\phi(x, t, y) = \langle x, y\rangle + \frac{1}{2} t\langle Ay, y\rangle + \mathcal{E}(x, t, y), \quad (x, t) \in \mathbb{R}^n \times \mathbb{R},$$

where A is the Hessian matrix $\partial^2_{yy}(\partial_t \phi)(0)$ and $\mathcal{E}$ satisfies

$$\mathcal{E}(x, t, y) = O\big(\big(|x| + |t|\big)^2|y|^2\big) + O\big(\big(|t| + |x|\big)|y|^3\big). \tag{3.1}$$

This already appears in [3,7] but we prove it here to clarify uniformity of phase functions which come up after translation and rescaling.

Proof. By Taylor's expansion in y,

$$\phi(x,t,y)=\phi(x,t,0)+\langle\partial_y\phi(x,t,0),y\rangle+\frac{1}{2}y^T\partial^2_{yy}\phi(x,t,0)y+R(x,t,y)$$

with $R(x,t,y)=O((|x|+|t|)|y|^3)$ because it can be assumed that $R(0,0,y)=0$ and terms in y or x,t only can be discarded. Also we can set $\phi(x,t,0)=0$, $\partial_y\phi(0)=0$, $\partial^2_{yy}\phi(0)=0$.

Since $\partial_y\phi(\cdot,t,0)$ is invertible, we can find the smooth map $I(x,t)$ such that

$$x=\partial_y\phi\bigl(I(x,t),t,0\bigr).$$

Making the change of variables $x\to I(x,t)$, we get

$$\phi(x,t,y)=\langle x,y\rangle+\frac{1}{2}y^T\partial^2_{yy}\phi\bigl(I(x,t),t,0\bigr)y+R\bigl(I(x,t),t,y\bigr).$$

By expansion in t, we observe that

$$\begin{aligned}\partial^2_{yy}\phi\bigl(I(x,t),t,0\bigr)=\partial^2_{yy}\phi\bigl(I(x,0),0,0\bigr)&+t\bigl[\partial_t\partial^2_{yy}\phi\bigl(I(x,0),0,0\bigr)\\&+\partial_x\partial^2_{yy}\phi\bigl(I(x,0),0,0\bigr)\partial_tI(x,0)\bigr]+O\bigl(t^2\bigr).\end{aligned}$$

Since $\partial_y\phi(0)=\partial_t(\partial_y\phi)(0)=0$, it is easy to see $I(x,t)=M^{-1}x+O(|t|^2+|xt|+|x|^2)$. Here $M=\partial^2_{xy}\phi(0)$. Hence it follows that

$$\begin{aligned}\phi(x,t,y)=x\cdot y&+\frac{1}{2}t\bigl[y^T\partial_x\partial^2_{yy}\phi(0)y\bigr]\bigl(M^{-1}x\bigr)+\frac{1}{2}ty^T\partial^2_{yy}\partial_t\phi(0)y\\&+O\bigl(|x|^2|y|^2+|t||x||y|^2+t^2|y|^2+|x||y|^3+|t||y|^3\bigr).\end{aligned}$$

Making the change of variables $y+\frac{1}{2}t(M^{-1})^T[y^T\partial_x\partial^2_{yy}\phi(0)y]^T\to y$ completes the proof because it only yields additional $O(t|y|^3)$-terms. □

Remark 3.2. By the elliptic condition (1.9) the matrix A in Lemma 3.1 has eigenvalues of the same sign and by further linear transforms we may assume A is the identity matrix.

Without loss of generality we may assume that

$$\operatorname{supp}a\subset B(0,\epsilon_0)\times Q(0,\epsilon_0)\subset\mathbb{R}^{n+1}\times\mathbb{R}^n,\quad 0<\epsilon_0\ll 1,$$

where $Q(0,\epsilon_0)$ is the cube centered at the origin with the side of the length ϵ_0. For each j we partition $Q(0,\epsilon_0)$ dyadically into $\sim 2^{nj}$ cubes $\{Q^j_\nu\}$ of side length 2^{-j} centered at ν. By a Whitney type decomposition of $Q(0,\epsilon_0)\times Q(0,\epsilon_0)$ away from its diagonal D, we have

$$Q\times Q\setminus D=\biggl(\bigcup_{\epsilon_0\geqslant 2^{-j}>\lambda^{-1/2}}\ \bigcup_{\operatorname{dist}(Q^j_\nu,Q^j_{\nu'})\sim 2^{-j}}Q^j_\nu\times Q^j_{\nu'}\biggr)\cup\biggl(\bigcup_{\operatorname{dist}(Q^{j_0}_\nu,Q^{j_0}_{\nu'})\leqslant 2^{-j_0}}Q^{j_0}_\nu\times Q^{j_0}_{\nu'}\biggr),$$

where j_0 is the largest integer satisfying $\lambda^{-1/2} \leqslant 2^{-j_0}$. This was used to derive linear estimates from bilinear ones (see [9,23,24]). For each $-\log \epsilon_0 \leqslant j \leqslant j_0$, we set

$$f_\nu^j = \chi_{Q_\nu^j} f.$$

Then we can write

$$\big(T_\lambda f(x,t)\big)^2 = \sum_{j_0 \geqslant j \geqslant -\log \epsilon_0} \sum_{\operatorname{dist}(Q_\nu^j, Q_{\nu'}^j) \sim 2^{-j}} T_\lambda\big(f_\nu^j\big)(x,t) T_\lambda\big(f_{\nu'}^j\big)(x,t).$$

When $j = j_0$, for simplicity we abused notation by saying $\operatorname{dist}(Q_\nu^{j_0}, Q_{\nu'}^{j_0}) \sim 2^{-j_0}$ to means $\operatorname{dist}(Q_\nu^{j_0}, Q_{\nu'}^{j_0}) \leqslant 2^{-j_0}$. Now it is enough to show that for the same p, q as in Theorem 1.3 and $\epsilon > 0$,

$$\Big\| \sum_{\operatorname{dist}(Q_\nu^j, Q_{\nu'}^j) \sim 2^{-j}} T_\lambda\big(f_\nu^j\big) T_\lambda\big(f_{\nu'}^j\big) \Big\|_{q/2} \leqslant C 2^{-\epsilon j} \lambda^{-\frac{2n+2}{q} + c\epsilon} \|f\|_p \|f\|_p. \tag{3.2}$$

There is a natural orthogonality among $\{T_\lambda(f_\nu^j) T_\lambda(g_{\nu'}^j)\}_{\operatorname{dist}(Q_\nu^j, Q_{\nu'}^j) \sim 2^{-j}}$ which is due to the fact that $\det(\partial_{xy}^2 \phi) \neq 0$. Precisely,

Lemma 3.3. *For $1 \leqslant p \leqslant 2$, $1 \leqslant r \leqslant \infty$ and $j \geqslant j_0$ and for $\epsilon > 0$,*

$$\Big\| \sum_{\operatorname{dist}(Q_\nu^j, Q_{\nu'}^j) \sim 2^{-j}} T_\lambda\big(f_\nu^j\big) T_\lambda\big(g_{\nu'}^j\big) \Big\|_p \leqslant C \lambda^\epsilon \Big(\sum_{\operatorname{dist}(Q_\nu^j, Q_{\nu'}^j) \sim 2^{-j}} \big\| T_\lambda\big(f_\nu^j\big) T_\lambda\big(g_{\nu'}^j\big) \big\|_p^p \Big)^{1/p} + \lambda^{-N} \|f\|_r \|g\|_r.$$

Using this it is enough to show that if $\operatorname{dist}(Q_\nu^j, Q_{\nu'}^j) \sim 2^{-j}$, then for $p \geqslant 2$, $q \geqslant 2(n+3)/(n+1)$,

$$\big\| T_\lambda\big(f_\nu^j\big) T_\lambda\big(f_{\nu'}^j\big) \big\|_{q/2} \leqslant C 2^{(\frac{2n+4}{q} - 2n(1-\frac{1}{p}) - \epsilon) j} \lambda^{-\frac{2n+2}{q} + c\epsilon} \big\| f_\nu^j \big\|_p \big\| f_{\nu'}^j \big\|_p. \tag{3.3}$$

(Note that one can assume that $n \geqslant 3$ by Hörmander's result [7] and $q/2 \leqslant 2$ by results due to Stein [17].) Indeed, by Lemma 3.3 and Schwarz's inequality,

$$\begin{aligned} \Big\| \sum_{\operatorname{dist}(Q_\nu^j, Q_{\nu'}^j) \sim 2^{-j}} T_\lambda\big(f_\nu^j\big) T_\lambda\big(f_{\nu'}^j\big) \Big\|_{q/2} &\leqslant C 2^{(\frac{2n+4}{q} - 2n(1-\frac{1}{p}) - \epsilon) j} \lambda^{-\frac{2n+2}{q} + c\epsilon} \\ &\quad \times \Big(\sum_\nu \big\| f_\nu^j \big\|_p^q \Big)^{1/q} \Big(\sum_\nu \big\| f_\nu^j \big\|_p^q \Big)^{1/q} + \lambda^{-N} \|f\|_r \|f\|_r. \end{aligned}$$

Hence we get (3.2) because $(2n+4)/q - 2n(1-1/p) \leqslant 0$ and we may assume $q \geqslant p$.

Proof of Lemma 3.3. Since a is compactly supported, it is enough to show Lemma 3.3 by freezing t. To say, for any $t \in (-\epsilon_0, \epsilon_0)$,

$$\Big\| \sum_{\mathrm{dist}(Q^j_\nu, Q^j_{\nu'}) \sim 2^{-j}} T_\lambda\big(f^j_\nu\big)(\cdot, t) T_\lambda\big(g^j_{\nu'}\big)(\cdot, t) \Big\|_p$$

$$\leqslant C\lambda^\epsilon \Big(\sum_{\mathrm{dist}(Q^j_\nu, Q^j_{\nu'}) \sim 2^{-j}} \big\| T_\lambda\big(f^j_\nu\big)(\cdot, t) T_\lambda\big(g^j_{\nu'}\big)(\cdot, t) \big\|_p^p \Big)^{1/p} + \lambda^{-N} \|f\|_r \|g\|_r. \tag{3.4}$$

Let us set $A_\mu(x) = \psi(2^j(x-\mu))$ with $\psi \in C_0^\infty(B(0,1))$ satisfying $\sum_{k \in \mathbb{Z}^n} \psi(\cdot - k) = 1$ and set

$$B^\mu_{\nu,\nu'} F(x) = A_\mu(x) \int e^{i\lambda \Phi(x,t,Y)} A(x,t,Y) b_{\nu,\nu'}(Y) F(Y)\, dY, \quad Y = (y, y'),$$

where $A(x,t,Y) = a(x,t,y)a(x,t,y')$, $b_{\nu,\nu'}(Y) = \chi_{Q^j_\nu}(y)\chi_{Q^j_{\nu'}}(y')$ and $\Phi(x,t,Y) = \phi(x,t,y) + \phi(x,t,y')$. We consider the oscillatory integral I^μ given by

$$I^\mu(\xi, t, Y) = \int e^{i\lambda\Phi(x,t,Y) - ix\xi} A(x,t,Y) A_\mu(x)\, dx.$$

It is easy to see that $|\lambda \partial_x \Phi(x,t,Y) - \xi| \geqslant |\lambda \partial_x \Phi(\mu, t, \nu, \nu') - \xi| - O(\lambda 2^{-j})$ if $x \in \operatorname{supp} A_\mu$ and $Y \in \operatorname{supp} b_{\nu,\nu'}$. By routine integration by parts we see that if

$$\big|\lambda \partial_x \Phi(\mu, t, \nu, \nu') - \xi\big| \geqslant C\lambda 2^{-j}$$

and $Y \in \operatorname{supp} b_{\nu,\nu'}$, then for any α and N,

$$\big|\partial^\alpha_\xi I^\mu(\xi, t, Y)\big| \leqslant C 2^{-nj} \left(\frac{|\lambda \partial_x \Phi'(\mu, t, \nu, \nu') - \xi|}{2^j} \right)^{-N}$$

for any $N > 0$. From this and trivial bounds $I^\mu(\xi) \leqslant C2^{-nj}$, it follows that if $Y \in \operatorname{supp} b_{\nu,\nu'}$, then for any α and $N > 0$

$$\big|\partial^\alpha_\xi I^\mu(\xi, t, Y)\big| \leqslant C 2^{-nj} \left(1 + \frac{|\lambda \partial_x \Phi(\mu, t, \nu, \nu') - \xi|}{\lambda 2^{-j}} \right)^{-N}. \tag{3.5}$$

Here we used the fact that $\lambda \geqslant 2^{2j}$. This means that the Fourier transform of $B^\mu_{\nu,\nu'} F$ is essentially supported in $\lambda 2^{-j}$-neighborhood of $\lambda \partial_x \Phi(\mu, t, \nu, \nu')$.

Let η be a smooth function supported in $B(0,1)$ and $\eta = 1$ on $B(0, 1/2)$ and for $\epsilon > 0$, let us set

$$\eta^\mu_{\nu,\nu'}(\xi) = \eta\left(\frac{\lambda \partial_x \Phi(\mu, t, \nu, \nu') - \xi}{\lambda^{1+\epsilon} 2^{-j}} \right).$$

So, using (3.5) it is not difficult to show that for $1 \leqslant p, r \leqslant \infty$ and any N

$$\big\| \big(I^{\mu} - \eta^{\mu}_{\nu,\nu'}(D)\big) B^{\mu}_{\nu,\nu'} \big\|_{r-p} \leqslant C\lambda^{-N}, \tag{3.6}$$

where I is the identity operator and $\eta^{\mu}_{\nu,\nu'}(D)$ is the Fourier multiplier operator defined by the multiplier $\eta^{\mu}_{\nu,\nu'}$. In fact,

$$\big(I - \eta^{\mu}_{\nu,\nu'}(D)\big) B^{\mu}_{\nu,\nu'} F(w) = \int K(w,t,Y) F(Y)\, dY,$$

where

$$K(x,t,Y) = (2\pi)^{-n} \int \big(I - \eta^{\mu}_{\nu,\nu'}(\xi)\big) I^{\mu}_{\nu,\nu'}(\xi,t,Y) e^{i\xi x}\, d\xi.$$

By a simple computation using (3.5) it is easy to see that for any N,

$$\big|K(x,t,Y)\big| \leqslant C\lambda^{-N}\big(1+|x|\big)^{-N} \big|b_{\nu,\nu'}(Y)\big|.$$

Hence by Schur's test and interpolation with trivial estimates we get (3.6).

Since the number of (ν,ν') with $\mathrm{dist}(Q^j_\nu, Q^j_{\nu'}) \sim 2^{-j}$ is at most $C2^{nj}$, from (3.6) it follows that for $1 \leqslant p \leqslant 2$, $1 \leqslant r \leqslant \infty$,

$$\bigg\| \sum_{\mathrm{dist}(Q^j_\nu, Q^j_{\nu'}) \sim 2^{-j}} B^{\mu}_{\nu,\nu'} F \bigg\|_p \leqslant \bigg\| \sum_{\mathrm{dist}(Q^j_\nu, Q^j_{\nu'}) \sim 2^{-j}} \eta^{\mu}_{\nu,\nu'}(D) B^{\mu}_{\nu,\nu'} F \bigg\|_p + C\lambda^{-N} \|F\|_r.$$

By this and the fact that the supports of A_μ are essentially disjoint,

$$\begin{aligned} \bigg\| \sum_{\mathrm{dist}(Q^j_\nu, Q^j_{\nu'}) \sim 2^{-j}} \sum_{\mu} B^{\mu}_{\nu,\nu'}(F) \bigg\|_p &\leqslant C\bigg(\sum_{\mu} \bigg\| \sum_{\mathrm{dist}(Q^j_\nu, Q^j_{\nu'}) \sim 2^{-j}} B^{\mu}_{\nu,\nu'}(F) \bigg\|_p^p \bigg)^{1/p} \\ &\leqslant C\bigg(\sum_{\mu} \bigg\| \sum_{\mathrm{dist}(Q^j_\nu, Q^j_{\nu'}) \sim 2^{-j}} \eta^{\mu}_{\nu,\nu'}(D) B^{\mu}_{\nu,\nu'}(F) \bigg\|_p^p \bigg)^{1/p} \\ &\quad + C\lambda^{-N} \|F\|_r. \end{aligned}$$

Since $\partial_x \phi_1(\mu,t,\cdot)$ is diffeomorphism and $\mathrm{dist}(Q^j_\nu, Q^j_{\nu'}) \sim 2^{-j}$, for each μ the set $\{\partial_x \Phi(\mu,t,\nu,\nu')\colon \nu,\nu', \mathrm{dist}(Q^j_\nu, Q^j_{\nu'}) \sim 2^{-j}\}$ is essentially 2^{-j}-separated, that is, only $O(1)$ elements are contained in a ball of radius 2^{-j}. (Note that $\partial_x \Phi(\mu,t,\nu,\nu') = 2\partial_x \phi(\mu,t,\nu) + O(2^{-j})$ and ν are separated by $\sim 2^{-j}$.) So, the supports of $\eta^{\mu}_{\nu,\nu'}$ are overlapping at most $O(\lambda^{n\epsilon})$ because it is contained in the ball centered at $\lambda \partial_x \Phi(\mu,t,\nu,\nu')$ with radius $\lambda^{1+\epsilon} 2^{-j}$. Hence by Plancherel's theorem and interpolation with trivial L^1 estimates we get for $1 \leqslant p \leqslant 2$

$$\bigg\| \sum_{\mathrm{dist}(Q^j_\nu, Q^j_{\nu'}) \sim 2^{-j}} \eta^{\mu}_{\nu,\nu'}(D) F \bigg\|_p \leqslant C\lambda^{c\epsilon} \bigg(\sum_{\mathrm{dist}(Q^j_\nu, Q^j_{\nu'}) \sim 2^{-j}} \big\|\eta^{\mu}_{\nu,\nu'}(D) F\big\|_p^p \bigg)^{1/p}.$$

Applying this to the above inequality, we get

$$\Big\| \sum_{\mathrm{dist}(Q_\nu^j, Q_{\nu'}^j)\sim 2^{-j}} \sum_\mu B_{\nu,\nu'}^\mu F \Big\|_p \leqslant C\lambda^{c\epsilon}\Big(\sum_\mu \sum_{\mathrm{dist}(Q_\nu^j, Q_{\nu'}^j)\sim 2^{-j}} \big\| \eta_{\nu,\nu'}^\mu(D) B_{\nu,\nu'}^\mu F\big\|_p^p \Big)^{1/p}$$
$$+ C\lambda^{-N}\|F\|_r.$$

Trivially, $\eta_{\nu,\nu'}^\mu(D)$ is bounded on L^p with uniform norm. Replacing F with $f\otimes g$, the right-hand side of (3.4) is bounded by

$$C\lambda^{c\epsilon}\Big(\sum_{\mathrm{dist}(Q_\nu^j, Q_{\nu'}^j)\sim 2^{-j}} \sum_\mu \big\| B_{\nu,\nu'}^\mu (f\otimes g)\big\|_p^p \Big)^{1/p} + C\lambda^{-N}\|f\|_r\|g\|_r$$

because $B_{\nu,\nu'}^\mu(f\otimes g)(x) = A_\mu(x) T_\lambda(f_\nu^j)(\cdot,t) T_\lambda(g_{\nu'}^j)(\cdot,t)$. Since the supports of A_μ are essentially disjoint,

$$\sum_\mu \big\| B_{\nu,\nu'}^\mu(f,g)\big\|_p^p \leqslant C\big\| T_\lambda\big(f_\nu^j\big)(\cdot,t) T_\lambda\big(g_{\nu'}^j\big)(\cdot,t)\big\|_p^p.$$

This proves (3.4). $\square$

It remains to prove (3.3). When $2^{2j}\sim\lambda$, it is easy to show. Since $\det(\partial_{xy}^2\phi)\neq 0$, from the generalized Hausdorff–Young's inequality [7,18], we have $\|T_\lambda f\|_2 \leqslant C\lambda^{-n/2}\|f\|_2$. We also have $\|T_\lambda f_\nu^j\|_\infty \leqslant C\lambda^{-n/2}\|f_\nu^j\|_\infty$ because $\lambda\sim 2^{2j}$. From interpolation between these two, we get, in particular,

$$\big\| T_\lambda\big(f_\nu^j\big) T_\lambda\big(f_{\nu'}^j\big)\big\|_{\frac{n+1}{n}} \leqslant C\lambda^{-n} \big\| f_\nu^j\big\|_{\frac{2n+2}{n}} \big\| f_{\nu'}^j\big\|_{\frac{2n+2}{n}}.$$

Interpolation between this and the trivial L^1–L^∞ estimates gives the required estimates (3.3) for all p,q satisfying $(n+2)/q \leqslant n(1-1/p)$, $q\geqslant (2n+2)/n$.

We turn to the case $2^{2j}\ll\lambda$. Since f_ν^j, $g_{\nu'}^j$ are supported on the sets of measure $O(2^{-nj})$, by Hölder's inequality it is enough to show that if $\mathrm{dist}(Q_\nu^j, Q_{\nu'}^j)\sim 2^{-j}$, then for $q\geqslant 2(n+3)/(n+1)$,

$$\big\| T_\lambda\big(f_\nu^j\big) T_\lambda\big(f_{\nu'}^j\big)\big\|_{q/2} \leqslant C\lambda^{-\frac{2(n+1)}{q}+c\epsilon} 2^{(\frac{2n+4}{q}-2n(1-\frac{1}{2})-\epsilon)j} \big\| f_\nu^j\big\|_2 \big\| f_{\nu'}^j\big\|_2. \tag{3.7}$$

By Lemma 3.1 with $A=I$ (also see Remark 3.2) and translation $(y\to\nu+y)$ for both $T_\lambda(f_\nu^j)$ and $T_\lambda(f_{\nu'}^j)$, it is possible to write the phase as

$$\phi(x,t,y) = \langle x,y\rangle + \frac{1}{2}t|y|^2 + \mathcal{E}_\nu(x,t,y) \tag{3.8}$$

on the small neighborhood of the origin where $\mathcal{E}_\nu$ is a smooth function satisfying (3.1) uniformly in ν. Indeed, from Lemma 3.1 and translation $(y\to y+\nu)$, the phase is equal to $\langle x,y\rangle + \langle x,\nu\rangle +$

$t\langle\nu, y\rangle + t|y|^2/2 + t|\nu|^2/2 + \mathcal{E}(x, t, y+\nu)$. We may discard $\langle x, \nu\rangle$, $t|\nu|^2/2$ and make change of variables $x + t\nu \to x$ so that the phase ϕ is changed to take the form $\phi(x, t, y) = \langle x, y\rangle + t|y|^2/2 + \mathcal{E}(x - \nu t, t, y+\nu)$. Expanding Q in y at ν,

$$\mathcal{E}(x, t, y+\nu) = \mathcal{E}(x, t, \nu) + \partial_y \mathcal{E}(x, t, \nu) y + \frac{1}{2}\sum_{|\alpha|=2} \partial_y^\alpha \mathcal{E}(x, t, \nu) y^\alpha + R_\nu(x, t, y),$$

where $R_\nu(x, t, y) = \frac{1}{2}\sum_{|\alpha|=3}\int_0^1 \partial_y^\alpha \mathcal{E}(x, t, sy+\nu) y^\alpha\, ds$. Discarding $\mathcal{E}(x, t, \nu)$ and making the change of variables $x + \partial_y \mathcal{E}(x, t, \nu) \to x$ again, in (3.8) we may set

$$\mathcal{E}_\nu(x, t, y) = \frac{1}{2}\sum_{|\alpha|=2} \partial_y^\alpha \mathcal{E}\big(T(x, t, \nu), t, \nu\big) y^\alpha + R_\nu\big(T(x, t, \nu), t, y\big), \tag{3.9}$$

where $T(x, t, \nu) = x - \partial_y \mathcal{E}(x, t, \nu) - \nu t$. Then we see that $\mathcal{E}_\nu$ satisfies (3.1) uniformly in ν because $\partial_y^\alpha \mathcal{E}(x - \partial_y \mathcal{E}(x, t, \nu) - \nu t, t, \nu) = O(|x| + |t|)$ uniformly from Lemma 3.1.

By rescalings $y \to 2^{-j} y$, the phase ϕ is changed to $2^{-j}\phi_j$ where

$$\phi_j(x, t, y) = \langle x, y\rangle + 2^{-(j+1)} t|y|^2 + 2^j \mathcal{E}_\nu\big(x, t, 2^{-j} y\big).$$

Now we consider an operator with phase function ϕ_j. Let us set

$$Tf(x, t) = \int e^{i2^{-j}\lambda\phi_j(x,t,y)} \tilde{a}(x, t, y) f(y)\, dy, \tag{3.10}$$

where $\tilde{a}$ is a smooth cutoff function supported in a neighborhood of the origin. By rescaling $y \to 2^{-j} y$ for both $T_\lambda(f_\nu^j)$ and $T_\lambda(f_{\nu'}^j)$, to show (3.7), it is sufficient to show that if $\operatorname{dist}(\operatorname{supp} f, \operatorname{supp} g) \sim 1$, then for $q \geqslant 2(n+3)/(n+1)$ and $\epsilon > 0$

$$\|Tf_1 Tf_2\|_{q/2} \leqslant C\lambda^{-\frac{2(n+1)}{q} + c\epsilon} 2^{(\frac{2n+4}{q} - \epsilon) j} \|f_1\|_2 \|f_2\|_2.$$

In case of linear phase (i.e. $\mathcal{E}_\nu = 0$) we can simply apply rescaling $t \to 2^j t$ to obtain a uniform phase in j but it does not work here because $\mathcal{E}_\nu$ may contain terms with high order in t. To get around this, we make further decomposition of T.

Let π be a smooth function such that $\sum_{k\in\mathbb{Z}^n} \pi(\cdot - k) = 1$ and its Fourier transform is supported in $B(0, 1)$. Let $\{Q\}$ be a collection of cubes of side length 2^{-j} which partitions of $\mathbb{R}^n$ and let a_Q be the affine map sending Q to the unit cube centered at the origin to Q. Then we set

$$\pi_Q = \pi \circ a_Q, \qquad \tilde{\pi}_Q = \pi_Q\big(\lambda^{-1} 2^j \cdot\big).$$

(Here π_Q is a smooth function essentially supported in Q.) We also set

$$F_i = \mathcal{F}^{-1} f_i, \quad i = 1, 2.$$

Then it follows that

$$Tf_i = \sum_Q T(\widehat{\tilde{\pi}_Q F_i}) = \sum_Q \int K(x, t, z)\pi_Q\big(\lambda^{-1} 2^j z\big) F_i(z)\, dz, \quad i = 1, 2,$$

where

$$K(x,t,z)=\int e^{i(2^{-j}\lambda\phi_j(x,t,y)-yz)}\tilde{a}(x,t,y)\,dy. \tag{3.11}$$

Since $\nabla_y(\phi_j-yz)=x-z+O(2^{-j})$, from integration by parts,

$$\left|K\left(x,t,\lambda 2^{-j}z\right)\right|\leqslant C\left(1+\lambda 2^{-j}|x-z|\right)^{-N}$$

for any N if $|x-z|\geqslant C2^{-j}$.

We may assume that a is supported in the ball of radius ϵ_0 centered at the origin. For $0<\epsilon\ll 1$ and for each Q, we denote by $\widetilde{Q}\subset\mathbb{R}^{n+1}$ the cube $\lambda^\epsilon Q\times[-\epsilon_0,\epsilon_0]$ where $\lambda^\epsilon Q$ has the same center and side length $\lambda^\epsilon 2^{-j}$. Since $\lambda\gg 2^{2j}$, it follows that

$$\left(1-\chi_{\widetilde{Q}}(x,t)\right)\left|K\left(x,t,\lambda 2^{-j}z\right)\pi_Q(z)\right|\leqslant C\lambda^{-N}$$

for any N. By rescaling $z\to\lambda 2^{-j}z$ it is easy to see that

$$\left|T(\widehat{\tilde{\pi}_Q F_1})T(\widehat{\tilde{\pi}_{Q'} F_2})\right|\leqslant\left|T(\widehat{\tilde{\pi}_Q F_1})\chi_{\widetilde{Q}}T(\widehat{\tilde{\pi}_{Q'} F_2})\chi_{\widetilde{Q}'}\right|+C\lambda^{-N}\|F_1\|_2\|F_2\|_2$$

because $|T(\widehat{\tilde{\pi}_Q F_i})|\leqslant C(\lambda 2^{-2j})^{n/2}\|F_i\|_2$, $i=1,2$. Since $\operatorname{supp}K(\cdot,z)$ is contained in a fixed compact set, we get

$$\|Tf_1Tf_2\|_{q/2}\leqslant\sum_{Q,Q'}\left\|T(\widehat{\tilde{\pi}_Q F_1})T(\widehat{\tilde{\pi}_{Q'} F_2})\right\|_{L^{q/2}(\widetilde{Q}\cap\widetilde{Q}')}+C\lambda^{-N}\|F_1\|_2\|F_2\|_2. \tag{3.12}$$

So, we are reduced to showing that

$$\sum_{Q,Q'}\left\|T(\widehat{\tilde{\pi}_Q F_1})T(\widehat{\tilde{\pi}_{Q'} F_2})\right\|_{L^{q/2}(\widetilde{Q}\cap\widetilde{Q}')}\leqslant C\lambda^{-\frac{2(n+1)}{q}+c\epsilon}2^{\frac{2n+4}{q}j+\epsilon j}\|F_1\|_2\|F_2\|_2.$$

(Recall $\lambda\geqslant 2^{2j}$.) For fixed Q, the number of Q' with $\widetilde{Q}\cap\widetilde{Q}'\neq\emptyset$ is $O(\lambda^{n\epsilon})$. Hence it is enough to show that for any 2^{-j}-cube Q_0,

$$\left\|T(\widehat{\tilde{\pi}_Q F_1})T(\widehat{\tilde{\pi}_{Q'} F_2})\right\|_{L^{q/2}(Q_0\times[-\epsilon_0,\epsilon_0])}\leqslant C\lambda^{-\frac{2(n+1)}{q}+\epsilon}2^{\frac{2n+4}{q}j+c\epsilon j}\|\widehat{\tilde{\pi}_Q F_1}\|_2\|\widehat{\tilde{\pi}_{Q'} F_2}\|_2.$$

It implies the required by Plancherel's theorem and Schwarz's inequality because the supports of $\tilde{\pi}_{Q'}$ are essentially disjoint.

Let x_0 be the center of Q_0 and make the change of variable $x\to x_0+2^{-j}x$ to change ϕ_j to $2^{-j}xy+x_0y+2^{-j-1}t|y|^2+2^j\mathcal{E}_\nu(x_0+2^{-j}x,t,2^{-j}y)$. The supports of $\widehat{\tilde{\pi}_Q F_1}$ and $\widehat{\tilde{\pi}_{Q'} F_2}$ are separated by the distance ~ 1 because $2^{2j}\lambda^{-1}\ll 1$ and $\operatorname{dist}(\operatorname{supp}f_1,\operatorname{supp}f_2)\sim 1$. Therefore it is enough to show that if f_1 and f_2 are supported in $B(0,1)$ and if $\operatorname{dist}(\operatorname{supp}f_1,\operatorname{supp}f_2)\sim 1$, then

$$\|\widetilde{T}_{(\lambda 2^{-2j})}f_1\cdot\widetilde{T}_{(\lambda 2^{-2j})}f_2\|_{q/2}\leqslant C\lambda^{-\frac{2n+2}{q}+c\epsilon}2^{\frac{4n+4}{q}j+\epsilon j}\|f_1\|_2\|f_2\|_2, \tag{3.13}$$

where the oscillatory integral $\widetilde{T}_\lambda$ is defined as T_λ by replacing ϕ with

$$\tilde{\phi}(x,t,y) = \langle x, y\rangle + t|y|^2/2 + 2^{2j}\mathcal{E}_\nu\big(x_0 + 2^{-j}x, t, 2^{-j}y\big)$$

and a smooth amplitude function supported in a neighborhood of the origin.

We apply Theorem 1.1 with $\phi_1 = \phi_2 = \tilde{\phi}$, $\lambda 2^{-2j}$ replacing λ and smooth bump functions a_1, a_2 adapted $B(0,\epsilon_0) \times \operatorname{supp} f_1$, $B(0,\epsilon_0) \times \operatorname{supp} f_2$, respectively. Since $\mathcal{E}_\nu$ satisfies (3.1) uniformly in ν and $|(x_0 + 2^{-j}x, t)| \leqslant \epsilon_0$, we see that

$$\partial_x\tilde{\phi}(x,t,y) = y + O\big(\epsilon_0|y|^2\big) + O\big(2^{-j}|y|^3\big),$$
$$\partial^2_{yx}\tilde{\phi}(x,t,y) = I + O(\epsilon_0) + O\big(2^{-j}\big),$$
$$\partial_t\tilde{\phi}(x,t,y) = |y|^2/2 + O\big(\epsilon_0|y|^2\big) + O\big(2^{-j}|y|^3\big).$$

Since $\partial_y q(x,t,\partial_x\tilde{\phi}(x,t,y)) = \partial_y\partial_t\tilde{\phi}(x,t,y)[\partial^2_{yx}\tilde{\phi}(x,t,y)]^{-1}$,

$$\partial_y q\big(x,t,\partial_x\tilde{\phi}(x,t,y)\big) = y + O\big(\epsilon_0|y|\big) + O\big(2^{-j}|y|\big).$$

So, it follows that if $(x,t,\xi_1) \in \operatorname{supp} f_1$ and $(x,t,\xi_2) \in \operatorname{supp} f_2$, then

$$\text{the LHS of (1.4)} = |\xi_1 - \xi_2|^2 + O(\epsilon_0) + O\big(2^{-j}\big) \sim 1 \tag{3.14}$$

as long as ϵ_0 is small enough and j is large because $|\xi_1 - \xi_2| \sim 1$. Note that we can assume j is large because $2^{-j} \leqslant \epsilon_0$. Therefore we get (3.13) from Theorem 1.1 with λ replaced by $\lambda 2^{-2j}$.

To finish the proof we need to show that the constant C in (3.13) is uniform along $\tilde{\phi}$. That is,

$$\|\widetilde{T}_\lambda f_1 \widetilde{T}_\lambda f_2\|_{q/2} \leqslant C\lambda^{-\frac{2n+2}{q}+\epsilon}\|f_1\|_2\|f_2\|_2,$$

with C, independent of x_0 and ν, j. By retracing the proof of Theorem 1.1 it is not hard to see that for each ϵ the constant C in (1.5) remain valid as long as the LHS of (1.4) is uniformly bounded away from zero and the phases are bounded in C^∞ uniformly. In fact, the estimates in Lemmas 2.2, 2.4, and 2.5 are stable under small perturbation of phase since these are mainly relying on integration by parts, and so is Lemma 2.6 provided the LHS of (2.30) is bounded away from zero uniformly. Using (3.9) one can easily see that $\tilde{\phi}$ converges to

$$\tilde{\phi}_\infty = \langle x, y\rangle + \frac{1}{2}t|y|^2 + \frac{1}{2}\sum_{|\alpha|=2}\partial_y^\alpha\mathcal{E}\big(T(x_0,t,\nu),t,\nu\big)y^\alpha \quad \text{in } C^\infty$$

as $j \to \infty$. Hence $\|\tilde{\phi}\|_{C^\infty} \leqslant C$, independent of x_0 and ν, j. This and (3.14) gives the required uniform bound for $\widetilde{T}_\lambda$.

3.2. Proof of Theorem 1.4

Since ϕ is smooth and homogeneous of degree one in y, by translation, rescaling and rotation we may assume that the support of a is contained in

$$B(0,\epsilon_0)\times B(e_n,\epsilon_0)\subset\mathbb{R}^{n+1}\times\mathbb{R}^n,\quad 0<\epsilon_0\ll 1.$$

As before, writing $z=(x,t)\in\mathbb{R}^n\times\mathbb{R}$, we may assume that θ is parallel to t direction. That is, $\partial_y(\partial_t\phi)(0,e_n)=0$ and $M=\partial^2_{xy}\phi(0,e_n)$ is an invertible matrix. Here e_n is the unit vector in y_n-direction. Under these assumptions we can get a normal form for the phase ϕ. We write $y=(\bar y,y_n)\in\mathbb{R}^{n-1}\times\mathbb{R}$.

Lemma 3.4. *If the phase function ϕ satisfies* (1.12) *with θ parallel to t-direction in a neighborhood of* $(0,e_n)\in\mathbb{R}^{n+1}\times\mathbb{R}^n$, *$\phi$ can be written as*

$$\phi(x,t,y)=\langle x,y\rangle+\frac{1}{2}t\bar y^T\partial_t\partial^2_{\bar y\bar y}\phi(0,e_n)(\bar y/y_n)+y_n\mathcal{E}(x,t,\bar y/y_n),$$

$(\bar y,y_n)\in\mathbb{R}^{n-1}\times\mathbb{R}$ *on a small neighborhood of* $(0,e_n)$ *and $\mathcal{E}$ satisfies*

$$\mathcal{E}(x,t,\bar y)=O\big(\big(|t|+|x|\big)^2|\bar y|^2+\big(|x|+|t|\big)|\bar y|^3\big).\tag{3.15}$$

Proof. The proof is similar to that of Lemma 3.1. By Taylor's expansion at the origin in $\bar y$, we may write

$$\phi(x,t,\bar y,1)=\big\langle\partial_{\bar y}\phi(x,t,e_n),\bar y\big\rangle+\frac{1}{2}\bar y^T\partial^2_{\bar y\bar y}\phi(x,t,e_n)\bar y+R(x,t,\bar y)$$

with $R(x,t,\bar y)=O(|x|+|t|)|\bar y|^3$ because we may assume $\phi(x,t,e_n)=R(0,0,\bar y)=0$. Since $\partial_y\phi(\cdot,t,e_n)$ is invertible, $\partial_y\phi(0,e_n)=0$ (as before it can be assumed) and $\partial_y(\partial_t\phi)(0,e_n)=0$, there is a smooth map $I(x,t)$ such that $x=\partial_y\phi(I(x,t),t,e_n)$ and $I(x,t)=M^{-1}x+O(|t|^2+|xt|+|x|^2)$. Hence, making change of variables $x\to I(x,t)$ and by Taylor's expansion in t, we have

$$\begin{aligned}\partial^2_{\bar y\bar y}\phi\big(I(x,t),t,e_n\big)&=\partial^2_{\bar y\bar y}\phi\big(I(x,0),0,e_n\big)+t\big(\partial_t\partial^2_{\bar y\bar y}\phi\big(I(x,0),0,e_n\big)\\&\quad+\partial_x\partial^2_{\bar y\bar y}\phi\big(I(x,0),0,e_n\big)\partial_tI(x,0)\big)+O\big(t^2\big).\end{aligned}$$

Since we may assume $\partial^2_{\bar y\bar y}\phi(0,e_n)=0$ and $I(0)=0$, it follows that

$$\begin{aligned}\phi(x,t,\bar y,1)&=x_n+\langle\bar x,\bar y\rangle+\frac{1}{2}\big[\bar y^T\partial_x\partial^2_{\bar y\bar y}\phi(0,e_n)\bar y\big]\big(M^{-1}x\big)+\frac{1}{2}t\bar y^T\partial^2_{\bar y\bar y}\partial_t\phi(0,e_n)\bar y\\&\quad+O\big(\big(|x|+|t|\big)^2|\bar y|^2+\big(|x|+|t|\big)|\bar y|^3\big).\end{aligned}$$

Making the change of variables $\bar y+(M^{-1})^T[\bar y^T\partial_x\partial^2_{\bar y\bar y}\phi(0,e_n)\bar y]^T/2\to\bar y$ and writing $\phi(x,t,y)=y_n\phi(x,t,\bar y/y_n,1)$ (note $y_n\sim 1$) complete the proof. □

Let C_0 be the cube $Q(0, 2\epsilon_0) \subset \mathbb{R}^{n-1}$, and for each j, $1 \leqslant 2^j \leqslant \lambda^{1/2}$, we partition C_0 dyadically into $\sim 2^{(n-1)j}$ cubes $\{C_\theta^j\}$ of side length 2^{-j} centered at θ. Then by the same decomposition as before we have

$$C_0 \times C_0 \setminus D = \left(\bigcup_{\epsilon_0 \geqslant 2^{-j} > \lambda^{-1/2}} \bigcup_{\operatorname{dist}(C_\theta^j, C_{\theta'}^j) \sim 2^{-j}} C_\theta^j \times C_{\theta'}^j \right) \cup \left(\bigcup_{\operatorname{dist}(C_\theta^{j_0}, C_{\theta'}^{j_0}) \leqslant 2^{-j_0}} C_\theta^{j_0} \times C_{\theta'}^{j_0} \right),$$

where D is the diagonal of $C_0 \times C_0$ and j_0 is the largest integer satisfying $\lambda^{-1/2} \leqslant 2^{-j_0}$. Let us set

$$f_\theta^j(\xi) = \chi_{C_\theta^j}(\bar{\xi}/\xi_n) f(\xi).$$

Then, if follows that

$$(T_\lambda f)^2 = \sum_{1 \leqslant j < j_0} \sum_{\operatorname{dist}(C_\theta^j, C_{\theta'}^j) \sim 2^{-j}} T_\lambda f_\theta^j T_\lambda f_{\theta'}^j.$$

Here we again abused the notation by saying $\operatorname{dist}(C_\theta^j, C_{\theta'}^j) \sim 2^{-j}$ to mean $\operatorname{dist}(C_\theta^j, C_{\theta'}^j) \leqslant 2^{-j}$ when $2^{2j} \sim \lambda$. For $1 \leqslant j \leqslant j_0$, we claim that for p, q as in Theorem 1.4

$$\Big\| \sum_{\operatorname{dist}(C_\theta^j, C_{\theta'}^j) \sim 2^{-j}} T_\lambda f_\theta^j T_\lambda f_{\theta'}^j \Big\|_{q/2} \leqslant C 2^{-\epsilon j} \lambda^{-\frac{2n+2}{q} + c\epsilon} \|f\|_p \|f\|_p. \tag{3.16}$$

3.2.1. Orthogonality among $T_\lambda f_\theta^j T_\lambda f_{\theta'}^j$, $\operatorname{dist}(C_\theta^j, C_{\theta'}^j) \sim 2^{-j}$

As in the proof of Theorem 1.3, one can show that for $1 \leqslant p \leqslant 2$, $1 \leqslant r \leqslant \infty$,

$$\Big\| \sum_{\operatorname{dist}(C_\theta^j, C_{\theta'}^j) \sim 2^{-j}} T_\lambda f_\theta^j T_\lambda g_{\theta'}^j \Big\|_p \leqslant C\lambda^\epsilon \Big(\sum_{\operatorname{dist}(C_\theta^j, C_{\theta'}^j) \sim 2^{-j}} \big\| T_\lambda f_\theta^j T_\lambda g_{\theta'}^j \big\|_p^p \Big)^{1/p} + \lambda^{-N} \|f\|_r \|g\|_r. \tag{3.17}$$

Proof. The proof is similar to that of Lemma 2.5. So we shall be brief. It is enough to show (3.4) for each t as before. Fixing t, we consider

$$B_{\theta,\theta'}^\mu F(x) = A_\mu(x) \int e^{i\lambda\Phi(x,t,Y)} A(x,t,Y) b_{\theta,\theta'}(Y) F(Y)\, dY, \qquad Y = (y, y'),$$

where $b_{\theta,\theta'}(Y) = \chi_{C_\theta^j}(y/|y|)\chi_{C_{\theta'}^j}(y'/|y'|)$, $A(x,t,Y) = a(x,t,y)a(x,t,y')$ and $\Phi(x,t,Y) = \phi(x,t,y) + \phi(x,t,y')$. Recall A_μ that is supported a disjoint cubes of side length 2^{-j} centered at μ. Let us set

$$I^\mu(\xi, t, Y) = \int e^{i(\lambda\Phi(x,t,y,y') - x\xi)} A(x,t,Y) A_\mu(x)\, dx.$$

If $|x-\mu| \leqslant 2^{-j}$, $y/|y| \in C_\theta^j$ and $y'/|y'| \in C_{\theta'}^j$,

$$\lambda \partial_x \Phi(x,t,y,y') - \xi = \lambda \partial_x \Phi\big(\mu,t,|y|\theta,|y'|\theta'\big) - \xi + O\big(2^{-j}\lambda\big).$$

Hence, by routine integration by parts, we see that if $y/|y|, y'/|y'| \in C_\theta^j, C_{\theta'}^j$, respectively, then for any α and N,

$$\big|\partial_\xi^\alpha I^\mu(\xi,t,Y)\big| \leqslant C2^{-nj}\left(1+\frac{|\lambda\partial_x\Phi(\mu,t,\rho\theta,\rho'\theta')-\xi|}{\lambda 2^{-j}}\right)^{-N}. \tag{3.18}$$

Here $\rho=|y|$, $\rho'=|y'|$. (Note that we only need to consider μ contained in $B(0,\epsilon)$.) If $\operatorname{dist}(C_\theta^j, C_{\theta'}^j) \sim 2^{-j}$, $\partial_\xi^\alpha I_{\theta,\theta'}^\mu(\cdot,t,y,y')$ is rapidly decaying outside of the set

$$\mathcal{L}_{\theta,\theta'}^\mu = \big\{\xi\colon \xi = \lambda\rho\partial_x\phi(\mu,t,\theta) + O\big(\lambda 2^{-j}\big),\ \rho\in(2-2\epsilon_0, 2+2\epsilon_0)\big\}$$

which is essentially a rectangle of size $\lambda \times \lambda 2^{-j} \times \cdots \times \lambda 2^{-j}$ centered at $2\lambda\partial_x\phi(\mu,t,\theta)$ with the major direction parallel to $\partial_x\phi(\mu,t,\theta)$.

Let $\eta_{\theta,\theta'}^\mu$ be a smooth function adapted to $\lambda^\epsilon\mathcal{L}_{\theta,\theta'}^\mu$ in natural way as before. Here $\lambda^\epsilon\mathcal{L}_{\theta,\theta'}^\mu$ is the set of dimension $\lambda^{1+\epsilon}\times(\lambda^{1+\epsilon}2^{-j})\times\cdots\times(\lambda^{1+\epsilon}2^{-j})$ obtained by dilating $\mathcal{L}_{\theta,\theta'}^\mu$ by factor λ^ϵ from the center point $2\lambda\partial_x\phi(\mu,t,\theta)$. Then, it is easy to see that for any N, $1\leqslant p,r\leqslant\infty$,

$$\big\|\big(I-\eta_{\theta,\theta'}^\mu(D)\big)B_{\theta,\theta'}^\mu F\big\|_p \leqslant C\lambda^{-N}\|F\|_r,$$

provided that $\operatorname{dist}(C_\theta^j, C_{\theta'}^j)\sim 2^{-j}$. From Lemma 3.4 we can see $\partial_x\phi(\mu,t,\theta)=\theta+O(|\theta|^2)$ near e_n. Hence one can easily see that overlapping among $\{\lambda^\epsilon\mathcal{L}_{\theta,\theta'}^\mu\colon \operatorname{dist}(C_\theta^j, C_{\theta'}^j)\sim 2^{-j}\}$ is at most $\lambda^{c\epsilon}$. Therefore we get for $1\leqslant p\leqslant 2$,

$$\Big\|\sum_{\operatorname{dist}(C_\theta^j,C_{\theta'}^j)\sim 2^{-j}}\eta_{\theta,\theta'}^\mu(D)F\Big\|_p \leqslant C\lambda^{c\epsilon}\bigg(\sum_{\operatorname{dist}(C_\theta^j,C_{\theta'}^j)\sim 2^{-j}}\big\|\eta_{\theta,\theta'}^\mu(D)F\big\|_p^p\bigg)^{1/p}. \tag{3.19}$$

Now the remaining is the same as in the proof Lemma 3.3. We omit the details. □

3.2.2. Re-scaling for $T_\lambda f_\theta^j T_\lambda g_{\theta'}^j$

To prove Theorem 1.4 for $n=2$ we only need show the L^4 estimate because the other estimates follow from interpolation with trivial L^∞–L^2 estimate and if $n\geqslant 3$ we may assume $q\leqslant 4$ by the result in [13].

Since $1\leqslant q/2\leqslant 2$, we can use (3.17) with p replaced by $q/2$. So, it is sufficient to show that if $\operatorname{dist}(C_\theta^j, C_{\theta'}^j)\sim 2^{-j}$, then for $p\geqslant 2$ and $q\geqslant 2(n+3)/(n+1)$,

$$\big\|T_\lambda f_\theta^j T_\lambda g_{\theta'}^j\big\|_{q/2} \leqslant C2^{(\frac{2n+2}{q}-2(n-1)(1-\frac{1}{p})+\epsilon)j}\lambda^{-\frac{2n+2}{q}+\epsilon}\big\|f_\theta^j\big\|_p\big\|g_{\theta'}^j\big\|_p. \tag{3.20}$$

In fact, putting this in (3.17) with $g=f$ and using Schwarz's inequality, we get

$$\Big\| \sum_{\operatorname{dist}(C_\theta^j, C_{\theta'}^j)\sim 2^{-j}} T_\lambda f_\theta^j T_\lambda f_{\theta'}^j \Big\|_{q/2} \leqslant C 2^{(\frac{2n+2}{q}-2(n-1)(1-\frac{1}{p})+\epsilon)j} \lambda^{-\frac{2n+2}{q}+\epsilon}$$

$$\times \Big(\sum_\theta \big\| f_\theta^j \big\|_p^q\Big)^{1/q} \Big(\sum_\theta \big\| f_\theta^j \big\|_p^q\Big)^{1/q} + C\lambda^{-N} \|f\|_p \|g\|_p.$$

Since C_θ^j are boundedly overlapping and $q \geqslant p$, it follows $(\sum_\theta \|f_\theta^j\|_p^q)^{1/q} \leqslant C\|f\|_p$. Therefore we get (3.16) for p, q as in Theorem 1.4.

The case $\lambda \sim 2^{2j}$ is almost trivial. Actually it can be shown that for p, q, $1/p + 1/q \leqslant 1$,

$$\big\| T_\lambda f_\theta^j T_\lambda g_{\theta'}^j \big\|_{q/2} \leqslant C\lambda^{-\frac{n+1}{q}-(n-1)(1-\frac{1}{p})} \big\| f_\theta^j \big\|_p \big\| g_{\theta'}^j \big\|_p.$$

By interpolation it follows from the estimates

$$\big\| T_\lambda f_\theta^j \big\|_\infty \leqslant C \big\| f_\theta^j \big\|_1, \qquad \big\| T_\lambda f_\theta^j \big\|_\infty \leqslant C\lambda^{(-n+1)/2} \big\| f_\theta^j \big\|_\infty \quad \text{and}$$
$$\big\| T_\lambda f_\theta^j \big\|_2 \leqslant C\lambda^{-n/2} \big\| f_\theta^j \big\|_2$$

(also the same estimates for $T_\lambda g_{\theta'}^j$). The second is easy because f_θ^j is supported in a set of measure $\sim 2^{-(n-1)j}$ and the third is due to generalized Hausdorff–Young's inequality [18, p. 377] since $\partial_{xy}^2 \phi \neq 0$.

Now we turn to the case $\lambda \gg 2^{2j}$. By Hölder's inequality we only need to show (3.20) with $p = 2$ since f_θ^j, $g_{\theta'}^j$ are supported in sets of measure $\sim 2^{-(n-1)j}$. By an additional change of variables for $\bar{y}$ it can be assumed that $\partial \bar{y}^2 \partial_t \phi(0, e_n) = I$ because of (1.12).

By Lemma (3.4) and changing of variables $\bar{y} \to \bar{y} + \theta y_n$ for both $T_\lambda f_\theta^j$ and $T_\lambda g_{\theta'}^j$ and $(\bar{x}, x_n) \to (\bar{x} - t\theta, x_n - t|\theta|^2)$ (also discarding harmless terms), we have

$$\phi(x, t, y) = \langle x, y\rangle + \frac{1}{2} t|\bar{y}|^2/y_n + y_n \mathcal{E}\big(\bar{x} - t\theta, x_n - t|\theta|^2, \bar{y}/y_n + \theta\big).$$

Expanding $\mathcal{E}(\bar{x} - t\theta, x_n - t|\theta|^2, \cdot)$ at θ in $\bar{y}$ and making the change of variables $x + (\partial_{\bar{y}} \mathcal{E}(x, t, \theta), \mathcal{E}(x, t, \theta)) \to x$, we may replace the above phase with

$$\phi(x, t, y) = \langle x, y\rangle + \frac{1}{2} t|\bar{y}|^2/y_n + \frac{y_n}{2} \sum_{|\alpha|=2} \partial_{\bar{y}}^\alpha \mathcal{E}\big(T(x, t, \theta), t, \theta\big)(\bar{y}/y_n)^\alpha$$
$$+ \frac{y_n}{2} \sum_{|\alpha|=3} \int_0^1 \partial_{\bar{y}}^\alpha \mathcal{E}\big(T(x, t, \theta), t, s\bar{y}/y_n + \theta\big)(\bar{y}/y_n)^\alpha \, ds, \tag{3.21}$$

where $T(x, t, \theta) = (\bar{x} - t\theta - \partial_{\bar{y}} \mathcal{E}(x, t, \theta), x_n - t|\theta|^2 - \mathcal{E}(x, t, \theta))$. Since $|\partial_{\bar{y}}^\alpha \mathcal{E}(T(x, t, \theta), t, \theta)| \leqslant C(|x| + |t|)$ by (3.15), we may set

$$\phi(x, t, y) = \langle x, y\rangle + \frac{1}{2} t|\bar{y}|^2/y_n + \mathcal{E}_\theta(x, t, \bar{y}/y_n)$$

with $\mathcal{E}_\theta$ satisfying (3.15) uniformly in θ. Then by rescalings $\bar{y} \to 2^{-j}\bar{y}$ (and by further decomposition in y if necessary) it is enough to show the following: if f_1, f_2 are supported in

$$S_1 = \{y\colon |\bar{y}/y_n - \bar{y}_1| \leqslant 1/4,\ |y_n - 1| \leqslant \epsilon_0\},$$
$$S_2 = \{y\colon |\bar{y}/y_n - \bar{y}_2| \leqslant 1/4,\ |y_n - 1| \leqslant \epsilon_0\},$$

respectively, and $|\bar{y}_1 - \bar{y}_2| \sim 1$, then for $q \geqslant 2(n+3)/(n+1)$ and $\epsilon > 0$,

$$\|Tf_1 Tf_2\|_{q/2} \leqslant C2^{(\frac{2n+2}{q}+\epsilon)j}\lambda^{-\frac{2n+2}{q}+\epsilon}\|f_1\|_2\|f_2\|_2,$$

where T is defined as in (3.10) with the phase

$$\phi_j(x,t,y) = \bar{x}\cdot\bar{y} + 2^j x_n y_n + 2^{-j}t|\bar{y}|^2/(2y_n) + 2^j y_n \mathcal{E}_\theta\big(x,t,2^{-j}\bar{y}/y_n\big)$$

and with a smooth $\tilde{a}$ supported in $B(0,\epsilon_0)\times B(e_n,1)$. This can be proven by a modification of the argument in the proof of Theorem 1.3.

We replace f_i with $F_i = \mathcal{F}^{-1}f_i$. Let $\{Q\}$ be a collection of cubes of size $(2^{-j})^{n-1}\times 2^{-2j}$ in $\bar{x}, x_n$ directions, respectively, which partition $\mathbb{R}^n$. Then it follows that

$$Tf_i = \sum_Q \int_{\mathbb{R}^n} K(x,t,z)\pi_Q\big(\lambda^{-1}2^j\bar{z},\lambda^{-1}z_n\big)F_i(z)\,dz, \quad i=1,2,$$

where $K(x,t,z)$ is given by (3.11). Since $\mathcal{E}_\theta$ satisfies (3.15), it is easy to see

$$\partial_{\bar{y}}\big(\lambda 2^{-j}\phi_j(x,t,y) - \lambda 2^{-j}\bar{y}\cdot\bar{z} - \lambda y_n z_n\big) = \lambda 2^{-j}(\bar{x}-\bar{z}) + O\big(\lambda 2^{-2j}\big),$$
$$\partial_{y_n}\big(\lambda 2^{-j}\phi_j(x,t,y) - \lambda 2^{-j}\bar{y}\cdot\bar{z} - \lambda y_n z_n\big) = \lambda(x_n - z_n) + O\big(\lambda 2^{-2j}\big).$$

Hence by integration by parts

$$\big|K\big(x,t,\lambda 2^{-j}\bar{z},\lambda z_n\big)\big| \leqslant \big(1+\lambda 2^{-j}|\bar{x}-\bar{z}|\big)^{-N}\big(1+\lambda|x_n - z_n|\big)^{-N}$$

if $|\bar{x}-\bar{z}| \geqslant C2^{-j}$ and $|x_n - z_n| \geqslant C2^{-2j}$.

By $\widetilde{Q}$ we denote the cubes $\lambda^\epsilon Q \times [-\epsilon_0,\epsilon_0]$. Then, it is easy to see that for any N

$$\big(1-\chi_{\widetilde{Q}}(x,t)\big)\big|K\big(x,t,\lambda 2^{-j}\bar{z},\lambda z_n\big)\pi_Q(z)\big| \leqslant C\lambda^{-N}.$$

Using this kernel estimates and rescaling, we get (3.12). Hence by the same argument as before (see the part below (3.12)) it is sufficient to show that for any $(2^{-j})^{n-1}\times 2^{-2j}$ cube Q_0,

$$\big\|T(\widehat{\tilde{\pi}_Q F_1})T(\widehat{\tilde{\pi}_{Q'} F_2})\big\|_{L^{q/2}(Q_0\times[-\epsilon_0,\epsilon_0])} \leqslant C\lambda^{-\frac{2(n+1)}{q}+\epsilon}2^{\frac{2n+2}{q}j+c\epsilon j}\|\widehat{\tilde{\pi}_Q F_1}\|_2\|\widehat{\tilde{\pi}_{Q'} F_2}\|_2,$$

where $\tilde{\pi}_Q = \pi_Q(\lambda^{-1}2^j\bar{z},\lambda^{-1}z_n)$.

Let x_0 be the center of Q_0 and change variables $x \to x_0 + (2^{-j}\bar{x}, 2^{-2j}x_n)$. Since the support of $\widehat{\tilde{\pi}_Q F_i}$ is contained in $S_i + O(2^{2j}\lambda^{-1})$ (note $2^{2j}\lambda^{-1} \ll 1$), the matters are reduced to showing the following: if f_1 and f_2 are supported in $S_1 + O(1/100)$ and $S_2 + O(1/100)$, respectively, then

$$\|\widetilde{T}_{(\lambda 2^{-2j})} f_1 \widetilde{T}_{(\lambda 2^{-2j})} f_2\|_{L^{q/2}(Q)} \leqslant C\big(\lambda 2^{-2j}\big)^{-\frac{2(n+1)}{q}+\epsilon} \|f_1\|_2 \|f_2\|_2, \tag{3.22}$$

where the oscillatory integral $\widetilde{T}_\lambda$ is defined as T_λ by smooth amplitude functions supported in a small neighborhood of $(0, e_n)$ and by replacing the phase ϕ with

$$\tilde{\phi}(x,t,y) = xy + t|\bar{y}|^2/(2y_n) + 2^{2j} y_n \mathcal{E}_\theta\big(x_0 + \big(2^{-j}\bar{x}, 2^{-2j}x_n\big), t, 2^{-j}\bar{y}/y_n\big).$$

To show (3.22) we apply Theorem 1.2 with $\phi_1 = \phi_2 = \tilde{\phi}$ and smooth bump functions a_1, a_2 adapted $B(0,1) \times \operatorname{supp} f_1$, $B(0,1) \times \operatorname{supp} f_2$, respectively. Since $\mathcal{E}_\theta$ satisfies (3.15) uniformly in θ and $|(x_0 + 2^{-j}x, t)| \leqslant \epsilon_0$,

$$\begin{gathered}
\partial_x \tilde{\phi}(x,t,y) = y + O\big(\epsilon_0 |y|^2\big) + O\big(2^{-j}|y|^3\big), \\
\partial^2_{yx} \tilde{\phi}(x,t,y) = I + O(\epsilon_0) + O\big(2^{-j}\big), \\
\partial_t \tilde{\phi}(x,t,y) = |y|^2/2y_n + O\big(\epsilon_0 |y|^2\big) + O\big(2^{-j}|y|^3\big).
\end{gathered}$$

Recalling $\partial_y q(x,t,\partial_x \tilde{\phi}(x,t,y)) = \partial_y \partial_t \tilde{\phi}(x,t,y)[\partial^2_{yx}\tilde{\phi}(x,t,y)]^{-1}$, we see

$$\partial_y q\big(x,t,\partial_x \tilde{\phi}(x,t,y)\big) = \big(\bar{y}/y_n, -|\bar{y}|^2/\big(2y_n^2\big)\big) + O\big(\epsilon_0|y|\big) + O\big(2^{-j}|y|\big).$$

Hence, if $\xi \in \operatorname{supp} f_1$ and $\eta \in \operatorname{supp} f_2$, then

$$\text{the LHS of (1.6)} = |\bar{\xi}/\xi_n - \bar{\eta}/\eta_n|^2 + O(\epsilon_0) + O\big(2^{-j}\big) \sim 1 \tag{3.23}$$

for $i = 1, 2$ as long as ϵ_0 is small enough and j is large enough. Therefore we get the required estimates (3.22) from Theorem 1.2 with $\lambda 2^{-2j}$.

In order to finish the proof we have to show that the constant C in (3.22) is uniform regardless of x_0 and ν, j. As before, from the proof of Theorem 1.2 it is not hard to see that for each fixed ϵ the bounds C remain under smooth perturbation stable as long as the left-hand side of (1.6) is bounded away from zero and the C^∞ norms of the phases are bounded uniformly. We see that $\|\tilde{\phi}\|_{C^\infty} \leqslant C$ with C, independent of x_0 and ν, j because $\tilde{\phi}$ converges to

$$\tilde{\phi}_\infty(x,t,y) = \langle x, y\rangle + \frac{1}{2}t|\bar{y}|^2/y_n + \frac{y_n}{2}\sum_{|\alpha|=2} \partial^\alpha_{\bar{y}} \mathcal{E}\big(T(x_0,t,\theta), t, \theta\big)(\bar{y}/y_n)^\alpha$$

as $j \to \infty$ (see (3.21)). Therefore from this and (3.23) we see (3.22) is valid uniformly in $\tilde{\phi}$.

3.3. Proof of Corollary 1.5

By the standard Littlewood–Paley decomposition and rescaling it is enough to show that for p, q as in Corollary 1.5 and $\epsilon > 0$,

$$\|\mathcal{F}_\lambda f\|_q \leqslant C\lambda^{\frac{n-1}{2} - \frac{n+1}{q} + \frac{1-n}{p} + \epsilon} \|f\|_p,$$

where

$$\mathcal{F}_\lambda f(x,t) = \int_{\mathbb{R}^n} e^{i\lambda\phi(x,t,\xi)} a(x,t,\xi)\beta\big(|\xi|\big)\widehat{f}(\xi)\,d\xi,$$

and β is a smooth function supported in $[1/2, 2]$. Here by a finite decomposition we may assume that a is supported in $B(0,\epsilon_0) \times B(e_n,\epsilon_0) \subset \mathbb{R}^{n+1} \times \mathbb{R}^n$, $0 < \epsilon_0 \ll 1$. Let us set

$$(1/p_0, 1/q_0) = \big((n^2+2n-3)\big/2\big(n^2+2n-1\big), \big(n^2-1\big)\big/2\big(n^2+2n-1\big)\big).$$

Since the Hessian matrix of $\phi(x,t,\cdot)$ has $n-1$ nonzero eigenvalues, L^∞–L^∞ and L^1–L^∞ estimates follow from the stationary phase method. It also can be shown using Lemma 3.5. Hence it is enough to show the case $(1/p, 1/q) = (1/p_0, 1/q_0)$.

First we show the case $n \geqslant 3$. We make use of the same decomposition in Section 3.2. Using C_θ^j, $-\log\epsilon_0 \leqslant j \leqslant j_0$, we write

$$(\mathcal{F}_\lambda f)^2 = \sum_{1\leqslant j<j_0} \sum_{\operatorname{dist}(C_\theta^j, C_{\theta'}^j)\sim 2^{-j}} \mathcal{F}_\lambda f_\theta^j \mathcal{F}_\lambda f_{\theta'}^j,$$

where

$$\widehat{f_\theta^j}(\xi) = \chi_{C_\theta^j}(\xi/\xi_n)\widehat{f}(\xi).$$

From the argument in the proof of (3.17), it is not hard to see that for $1 \leqslant q/2 \leqslant 2$, $2 < r \leqslant q < \infty$ and $\epsilon > 0$,

$$\begin{aligned} &\Big\| \sum_{\operatorname{dist}(C_\theta^j, C_{\theta'}^j)\sim 2^{-j}} \big(\mathcal{F}_\lambda f_\theta^j \mathcal{F}_\lambda f_{\theta'}^j\big) \Big\|_{q/2} \\ &\quad \leqslant C\lambda^\epsilon \bigg(\sum_{\operatorname{dist}(C_\theta^j, C_{\theta'}^j)\sim 2^{-j}} \big\|\mathcal{F}_\lambda f_\theta^j \mathcal{F}_\lambda f_{\theta'}^j\big\|_{q/2}^{q/2} \bigg)^{2/q} + \lambda^{-N}\|f\|_r^2. \end{aligned} \tag{3.24}$$

In fact, for $\operatorname{dist}(C_\theta^j, C_{\theta'}^j) \sim 2^{-j}$ let us set

$$\widetilde{B}_{\theta,\theta'}^\mu F(x) = A_\mu(x) \int e^{i\lambda\Phi(x,t,\Xi)} A(x,t,\Xi) b_{\theta,\theta'}(\Xi)\widehat{F}(\Xi)\,d\Xi, \quad \Xi = (\xi,\xi'),$$

where $b_{\theta,\theta'}(\Xi) = \chi_{C_\theta^j}(\xi/|\xi|)\chi_{C_{\theta'}^j}(\xi'/|\xi'|)$, $A(x,t,\Xi) = a(x,t,\xi)a(x,t,\xi')\beta(|\xi|)\beta(|\xi'|)$ and $\Phi(x,t,\Xi) = \phi(x,t,\xi) + \phi(x,t,\xi')$. Then by the estimates like (3.18) we can see that for $1 \leqslant r, p \leqslant \infty$ and $N > 0$,

$$\big\|\big(I - \eta_{\theta,\theta'}^\mu(D)\big)\widetilde{B}_{\theta,\theta'}^\mu F\big\|_p \leqslant C\lambda^{-N}\|F\|_r. \tag{3.25}$$

Similarly we can see that the Fourier transforms of $\widetilde{B}_{\theta,\theta'}^\mu F$ are essentially supported in the set $\mathcal{L}_{\theta,\theta'}^\mu$. Hence, using (3.19) and similar argument as before we get (3.24).

Therefore by (3.24) it is sufficient to show that if $\mathrm{dist}(C_\theta^j, C_{\theta'}^j) \sim 2^{-j}$, for $\epsilon > 0$

$$\big\|\mathcal{F}_\lambda f_\theta^j \mathcal{F}_\lambda f_{\theta'}^j\big\|_{q/2} \leqslant C\lambda^{(n-1)(1-\frac{2}{p})-\frac{2n+2}{q}+c\epsilon}2^{2j(\frac{n+1}{q}+\frac{n-1}{p}-(n-1))}\big\|f_\theta^j\big\|_p\big\|f_{\theta'}^j\big\|_p. \tag{3.26}$$

We get the required estimate by summation along j after plugging this in the above, using Schwarz's inequality and $(\sum_\theta \|f_\theta^j\|_p^q)^{1/q} \leqslant \|f\|_p$ which is valid as long as $2 \leqslant p \leqslant q < \infty$.

Since the line segment $[(0,0),(1/2,(n+1)/2(n+3))]$ and $(n+1)/q = (n-1)(1-1/p)$ intersect each other at $(1/p,1/q) = (1/p_0,1/q_0)$, it is sufficient to show (3.26) when $(1/p,1/q)$ is contained in the line segment $[(0,0),(1/2,(n+1)/2(n+3))]$. By interpolation we only need to show the cases $(p,q) = (\infty,\infty)$, $(p,q) = (2,2(n+3)/(n+1))$. But the case $(p,q) = (2,2(n+3)/(n+1))$ follows from (3.20) by Plancherel's theorem.

Lemma 3.5. *If $\widehat{f}$ is supported on a sector of angle $O(\lambda^{-1/2})$ which is contained in $\{\xi\colon |\xi| \sim 1\}$, then*

$$\|\mathcal{F}_\lambda f\|_\infty \leqslant C\|f\|_\infty.$$

Proof. It can be shown by direct kernel estimate. By rotation we may assume that $\widehat{f}$ is supported in a $\lambda^{-1/2}$ angular sector centered at e_n. Let $\psi \in C_0(\mathbb{R}^{n-1})$ with $\psi = 1$ on a neighborhood of the origin. Then we may write $\mathcal{F}_\lambda f = \int K(x,t,y)f(y)\,dy$ where

$$K(x,t,y) = (2\pi)^{-n}\int\limits_{\mathbb{R}^n} e^{i(\lambda\phi(x,t,\xi)-y\xi)}a(x,t,\xi)\beta\big(|\xi|\big)\psi\big(\lambda^{1/2}\bar{\xi}/\xi_n\big)\,d\xi.$$

Obviously, it enough to show that

$$\big|K(x,t,y)\big| \leqslant C\lambda^{-(n-1)/2}\big(1+\lambda^{-1/2}|\lambda\bar{x}-\bar{y}|+|\lambda x_n - y_n|\big)^{-N}.$$

Using Lemma 3.4, we see that the phase part

$$\lambda\phi\big(x,t,\lambda^{-1/2}\bar{\xi},\xi_n\big) - \big\langle\big(\lambda^{-1/2}\bar{y},y_n\big),\xi\big\rangle = \lambda^{-1/2}\langle\lambda\bar{x}-\bar{y},\bar{\xi}\rangle + (\lambda x_n - y_n)\xi_n + \mathcal{E}(x,t,\xi),$$

where $\partial_\xi^\alpha\mathcal{E} = O(1)$. By making the change of variables $\xi \to (\lambda^{-1/2}\bar{\xi},\xi_n)$ and routine integration by parts we get the required estimate. $\square$

Decomposing f_θ^j into at most $C2^{-j}\lambda^{1/2}$ functions which are Fourier supported in such $O(\lambda^{-1/2})$-sectors, by the above lemma we get $\|\mathcal{F}_\lambda f_\theta^j\|_\infty \leqslant C2^{-(n-1)j}\lambda^{(n-1)/2}\|f_\theta^j\|_\infty$ (and $\|\mathcal{F}_\lambda f_{\theta'}^j\|_\infty \leqslant C2^{-(n-1)j}\lambda^{(n-1)/2}\|f_{\theta'}^j\|_\infty$). This gives (3.26) for $(p,q) = (\infty,\infty)$.

Now we prove the remain case $n = 2$. Interpolation between $l^2(L^2)$–L^2 (3.19), and trivial $l^1(L^\infty)$–L^∞ estimates, it follows that for $2 \leqslant p \leqslant \infty$,

$$\Big\|\sum_{\mathrm{dist}(C_\theta^j,C_{\theta'}^j)\sim 2^{-j}} \eta_{\theta,\theta'}^\mu(D)F\Big\|_p \leqslant C\lambda^{c\epsilon}\Big(\sum_{\mathrm{dist}(C_\theta^j,C_{\theta'}^j)\sim 2^{-j}} \big\|\eta_{\theta,\theta'}^\mu(D)F\big\|_p^{p'}\Big)^{1/p'}.$$

Using this and (3.25), it is easy to see that for $2 \leqslant q/2 < \infty$ and $r \leqslant q$,

$$\Big\| \sum_{\mathrm{dist}(C_\theta^j, C_{\theta'}^j) \sim 2^{-j}} \mathcal{F}_\lambda f_\theta^j \mathcal{F}_\lambda f_{\theta'}^j \Big\|_{q/2} \leqslant C\lambda^\epsilon \Big(\sum_{\mathrm{dist}(C_\theta^j, C_{\theta'}^j) \sim 2^{-j}} \big\| \mathcal{F}_\lambda f_\theta^j \mathcal{F}_\lambda f_{\theta'}^j \big\|_{q/2}^{(q/2)'} \Big)^{1/(q/2)'} + \lambda^{-N} \|f\|_r^2.$$

By plugging (3.26) in the above and Schwarz's inequality, we see that the left-hand side is bounded by

$$C\lambda^{1-\frac{6}{q}+\frac{2}{p}+\epsilon} 2^{j(\frac{3}{p}+\frac{1}{p}-1)} \Big(\sum \|f_\theta^j\|_p^{2(q/2)'} \Big)^{\frac{1}{2(q/2)'}} \Big(\sum \|f_{\theta'}^j\|_p^{2(q/2)'} \Big)^{\frac{1}{2(q/2)'}} + C\lambda^{-N} \|f\|_p \|f\|_p.$$

Note that $p \leqslant 2(q/2)'$ when $(p,q) = (p_0, q_0)$. Hence $(\sum \|f_\theta^j\|_p^{2(q/2)'})^{\frac{1}{2(q/2)'}} \leqslant C\|f\|_p$. This completes the proof.

References

[1] B. Barcelo, On the restriction of the Fourier transform to a conical surface, Trans. Amer. Math. Soc. 292 (1985) 321–333.

[2] J. Bourgain, Besicovitch type maximal operators and applications to Fourier analysis, Geom. Funct. Anal. 1 (1991) 147–187.

[3] J. Bourgain, L^p-estimates for oscillatory integrals in several variables, Geom. Funct. Anal. 1 (4) (1991) 321–374.

[4] J. Bourgain, Estimates for cone multipliers, in: Geometric Aspects of Functional Analysis, in: Oper. Theory Adv. Appl., vol. 77, Birkhäuser, Basel, 1995, pp. 41–60.

[5] J. Bourgain, Some new estimates on oscillatory integrals, in: Essays on Fourier Analysis in Honor of Elias M. Stein, Princeton Univ. Press, Princeton, NJ, 1995, pp. 83–112.

[6] L. Carleson, P. Sjölin, Oscillatory integrals and a multiplier problem for the disc, Studia Math. 44 (1972) 287–299.

[7] L. Hörmander, Oscillatory integrals and multipliers on FL^p, Ark. Mat. 11 (1973) 1–11.

[8] I. Łaba, T. Wolff, A local smoothing estimate in higher dimensions, J. Anal. Math. 88 (2002) 149–171.

[9] S. Lee, Improved bounds for Bochner–Riesz and maximal Bochner–Riesz operators, Duke Math. J. 122 (2004) 105–235.

[10] S. Lee, Bilinear restriction estimates for surfaces with curvatures of different signs, Trans. Amer. Math. Soc. 358 (2006) 3511–3533.

[11] W. Minicozzi, C.D. Sogge, Negative results for Nikodym maximal functions and related oscillatory integrals in curved space, Math. Res. Lett. 4 (1997) 221–237.

[12] G. Mockenhaupt, Bounds in Lebesgue spaces of oscillatory integral operators, Habilitationsschrift, Universität Siegen, 1996.

[13] G. Mockenhaupt, A. Seeger, C. Sogge, Local smoothing of Fourier integral operators and Carleson–Sjölin estimates, J. Amer. Math. Soc. 6 (1993) 60–135.

[14] W. Schlag, C.D. Sogge, Local smoothing estimates related to the circular maximal theorem, Math. Res. Lett. 4 (1997) 1–15.

[15] C.D. Sogge, Propagation of singularities and maximal functions in the plane, Invent. Math. 104 (1991) 349–376.

[16] C.D. Sogge, Fourier Integrals in Classical Analysis, Cambridge Tracts in Math., vol. 105, Cambridge Univ. Press, Cambridge, 1993.

[17] E. Stein, Oscillatory integrals in Fourier analysis, in: Beijing Lectures in Harmonic Analysis, in: Ann. of Math. Stud., vol. 112, Princeton Univ. Press, Princeton, NJ, 1986, pp. 307–355.

[18] E. Stein, Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals, Princeton Univ. Press, Princeton, NJ, 1993.

[19] T. Tao, Endpoint bilinear restriction theorems for the cone and some sharp null from estimates, Math. Z. 238 (2001) 215–268.

[20] T. Tao, A sharp bilinear restriction estimate for paraboloids, Geom. Funct. Anal. 13 (2003) 1359–1384.
[21] T. Tao, Recent progress on the restriction conjecture, preprint.
[22] T. Tao, A. Vargas, A bilinear approach to cone multipliers I. Restriction estimates, Geom. Funct. Anal. 10 (2000) 185–215.
[23] T. Tao, A. Vargas, A bilinear approach to cone multipliers II. Application, Geom. Funct. Anal. 10 (2000) 216–258.
[24] T. Tao, A. Vargas, L. Vega, A bilinear approach to the restriction and Kakeya conjecture, J. Amer. Math. Soc. 11 (1998) 967–1000.
[25] A. Vargas, Restriction theorems for a surface with negative curvature, Math. Z. 249 (2005) 97–111.
[26] L. Wisewell, Kakeya sets of curves, Geom. Funct. Anal. 15 (2005) 1319–1326.
[27] T. Wolff, Recent work connected with the Kakeya problem, in: Prospects in Mathematics, Princeton, NJ, 1996, Amer. Math. Soc., Providence, RI, 1999, pp. 129–162.
[28] T. Wolff, Local smoothing type estimates on L^p for large p, Geom. Funct. Anal. 10 (2000) 1237–1288.
[29] T. Wolff, A sharp cone restriction estimate, Ann. of Math. (2) 153 (2001) 661–698.